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Fluid model with jumps in heavy traffic

M. Dimitrov*

* *Department of mathematics,*
University of National and World Economy,
Students campus, Sofia 1700, Bulgaria

Abstract. In the conventional Markov modulated fluid queue, the buffer content process has a continuous sample path. This paper considers a general stochastic fluid model with a single infinite capacity buffer whose buffer content process may have jumps. The continuous, as well as the instantaneous, change is modulated by an external environment process as a finite state continuous time Markov chain. Heavy traffic limit theorem for the distribution of the stationary fluid content is considered.

Keywords: single-server queue, fluid model with jumps, heavy traffic limit theorem.

1. Introduction

Classical fluid queueing models are queueing models in which work enters and leaves a buffer non-instantaneously, i.e. like a fluid. They are used for modeling and performance analysis of high-speed data networks, and large scale production systems.

In the present study we used the analytical approach introduced in [2] and prove a heavy traffic limit theorem for steady state fluid content for fluid model with jumps, considered in [1].

2. The model

We will continue the investigation of a fluid model introduced in [1]. The buffer content $X(t)$ can increase continuously and also by instantaneous jumps. The rate of change of the fluid level depends on the state of an external stochastic process $\{I(t), t \geq 0\}$, with a finite space $S = \{0, 1, \dots, K\}$. The buffer content increases continuously with rate $c_i \in (-\infty, +\infty)$ if the process $\{I(t), t \geq 0\}$ is in a state i. The process $\{I(t), t \geq 0\}$ remains an exponential amount of time at each state $i \in S$ with a parameter λ_i and, when leaving such state i, it will make a jump from state i to state j, not necessarily different from i with probability p_{ij}. When the $I(t)$ process jumps from state i to state j the amount of fluid in the buffer can increase by a non-negative random amount with a given c.d.f, $G_{ij}(y), y \geq 0$ with mean m_{ij} and second moment v_{ij}. A bivariate process $\{X(t), I(t)\}$ can jump from state (x, i) to state $(x + y, j)$ with rate $\lambda_i p_{ij} dG_{ij}(y), y \geq 0$. We need some notations $C =$ diag $(c_0, c_1, \dots, c_K)$, $P = ||p_{ij}||, 0 \leq i, j \leq K$, $M = ||m_{ij}||, 0 \leq i, j \leq K$,

$V=||v_{ij}||, 0 \le i,j \le K$, $\Lambda = \text{diag}\ (\lambda_0, \lambda_1, \dots, \lambda_K)$, $\lambda_i = \lambda\lambda_i^0$, $\Lambda^0 = \text{diag}\ \left(\lambda_0^0, \lambda_1^0, \dots, \lambda_K^0\right)$, $\lambda_i^0 > 0, 0 \le i \le K$, $\Lambda = \lambda\Lambda^0$, $\Gamma = \Lambda P \circ M$, $\Gamma^0 = \lambda_0\Lambda^0 P \circ M.$, $Q_{ij}(x) = \lambda_i p_{ij} G_{ij}(x), x \ge 0, i,j \in S$, $Q_{ii}(x) = \lambda_i P_{ii} G_{ii}(x), x \ge 0, i \in S$, $Q(x) = ||Q_{ij}||, i,j \in S$, $\overline{Q}_{ij}(s) = \int_0^{+\infty} e^{-sx}\, dQ_{ij}(x)$, $\overline{G}_{ij}(s) = \int_0^{+\infty} e^{-sx}\, dG_{ij}(x)$, $\overline{Q}(s) = ||\overline{Q}_{ij}(s)||$, $\overline{G}(s) = ||\overline{G}_{ij}(s)||$, $0 \le i,j \le K$, $\overline{Q}(s) = \Lambda P \circ \overline{G}(s)$.

Obviously, the process $I(t), t \ge 0$ is a continuous Markov chain on S with rate matrix $Q = Q(\infty)$. The matrix Q is defined as $Q = ||q_{ij}||, 0 \le i,j \le K$, $q_{ij} = \lambda_i p_{ij}, i \ne j$, $q_{ii} = \lambda_{ii} p_{ii} - \lambda_i$. Further we will assume that the process $I(t)$ is irreducible with the limiting distribution $\pi_i = \lim_{t\to+\infty} P(I(t) = i)$. The queueing system is stable iff

$$\sum_{i=0}^{K} \pi_i (c_i + \sum_{j=0}^{K} \lambda_i p_{ij} m_{ij}) = \pi(C + \Lambda P \circ M)e = \pi(C + \Gamma)e < 0.$$

The matrix operation $A \circ B$ denotes the matrix with (i,j)-th element is equal to $a_{ij}b_{ij}$. If the stability condition holds there exists a limiting distribution

$$F_i(x) = \lim_{t\to+\infty} P(X(t) \le x, I(t) = i),$$

$x \ge 0, i \in S$ of the bivariate stochastic process $\{X(t), I(t)\}$.

The LST $\phi(s)$ of the vector $F(X) = (F_0(x), F_1(x), \dots, F_K(x))$

$$\phi(s) = sF(0)C(sC - \overline{Q}(s))^{-1} \tag{1}$$

is derived in [1].

We shall assume from now that $\Lambda = \lambda\Lambda^0$. As the main process we shall consider the fluid content $X(t)$ in steady state. Our goal is to find the asymptotic behavior of the process $X(t)$ under heavy traffic condition when $\epsilon = \pi(C + \lambda\Gamma^0)e \to 0$. To avoid unnecessary complication with minor details we consider the following specific problem settings: all parameters except for λ are fixed, and λ increases in such a way that $\epsilon \to 0$, or $\lambda \to \lambda_0$ and $\lambda_0 = -\pi C e / \pi\Gamma^0 e$.

The set of equations with respect to unknown $a_{0,} a_1 \dots, a_K$:

$$Qa = (\pi(C + \Gamma)e)e - (C + \Gamma)e \tag{2}$$

always has a solution, since the stationary distribution π is orthogonal to the right side of this equation . We will define a matrix A and a matrix R. The matrix A is defined by the rows and columns with numbers $1, 2, \cdots, K$ of the matrix Q. The first row and the first column of matrix R are equal to zero vectors, and next K rows and K columns form matrix A^{-1}. Then a matrix QR looks as follows: the first raw is $(0, -\pi_1/\pi_0, \cdots, -\pi_K/\pi_0)$, and the first column is a zero vector, and the next elements form the identity matrix, and for any vector $x = (x_0, x_1, \cdots, x_k)$ we have

$$xQR = x - x_0\pi/\pi_0. \tag{3}$$

The next equation

$$(a_0, a_1, \cdots, a_K) = R((\pi(C+\Gamma)e)e - (C+\Gamma)e) \tag{4}$$

define the solution to the equation (2) when $a_0 = 0$.

3. The main results

The next theorem gives the mean value of a fluid level in steady state.

Theorem 1 *The mean value fluid level EX in steady state is given by the formula $EX = (\pi\Lambda P \circ Ve/2 + \pi\,(C+\Gamma)\,a - F(0)Ca)\,/\,(-\pi\,(C+\Gamma)\,e)$ where a is a solution to (2).*

Proof. Let us consider the LST $\phi(s)$ of the vector

$$(F_0(x), F_1(x), \cdots, F_K(x)) = F(x)$$

defined by the equation (1) $\phi(s)(sC - \overline{Q}(s)) = sF(0)C$. After some algebraic manipulations of equation (1) we get the main equation for the next study.

$$\phi(s)Q = \phi(s)(sC + \Lambda P \circ (G - \overline{G}(s))) - sF(0)C, \tag{5}$$

where $G = \overline{G}(0)$ and multiplying both sides of the equation (5) from the right by the vector $e = (1, 1, \ldots, 1)^T$), we have

$$F(0)Ce = \phi(s)\left(C + \Lambda P \circ \frac{G - \overline{G}(s)}{s}\right)e \tag{6}$$

After we let $s \to 0$, and taking into account that $\phi(0) = \pi$, and

$$\lim_{s\to 0}\left(G - \overline{G}(s)\right)/s = M \text{ we get } F(0)Ce = \pi\,(C+\Gamma)\,e.$$

Now, differentiating (6) with respect to s at the point $s = 0$ we have

$$(EX_0, EX_1, \ldots, EX_K)(C+\Gamma)e = -\frac{\pi\Lambda P \circ Ve}{2}, \tag{7}$$

where $EX_i = E(X(t), I(t) = i)$ in steady state, $i = 0, 1, \ldots, K$. Multiplying both sides of (5) by the vector a which is a solution to (2) we obtain

$$\phi(s)Qa = s\phi(s)\left(C + \Lambda P \circ \frac{G - \overline{G}(s)}{s}\right)a - sF(0)Ca.$$

After differentiating this equation and setting $s = 0$ we get

$$(EX_0, EX_1, \ldots, EX_K)Qa = -\pi\,(C + \Gamma)\,a + F(0)Ca \qquad (8)$$

Now, summing (7) and (8) we obtain EX. Thus, the theorem is proved.

Theorem 2 *Under heavy traffic assumptions, the random variables $-\epsilon X(t)$ and $I(t)$ are asymptotically independent and random variable $-\epsilon X(t)$ is asymptotically exponential with the mean*

$$N = (\pi\lambda_0\Lambda^0 P \circ V)/2 + \pi(C + \lambda_0\Lambda^0 P \circ M)a^0,$$

where a^0 is a solution to the equation (2) *at $\lambda = \lambda_0$.*

Proof: In order to find the asymptotic behavior of the stationary distribution of the process $\{X(t), I(t)\}$ we will consider the main equation (5). From (5) follows the next equation

$$\phi(s)(sC + \Lambda P \circ (G - \overline{G}(s)))e - s\epsilon = 0. \qquad (9)$$

From (5) , multiplying it by the matrix R, which was introduced earlier, we get the equation

$$\phi(s)QR = \phi(s)(sC + \Lambda P \circ (G - \overline{G}(s)))R - sF(0)CR. \qquad (10)$$

From (3), it follows $\phi(s)QR = \phi(s) - \pi\phi_0(s)/\pi_0$. Then the equation (10) could be written in the following form

$$\phi(s) = \pi\phi_0(s)/\pi_0 + \phi(s)(sC + \Lambda P \circ (G - \overline{G}(s)))R - sF(0)CR. \qquad (11)$$

Substituting $\phi(s)$ from (11) for $\phi(s)$ in the right hand side of the equation (11) we obtain

$$\phi(s) = \frac{1}{\pi_0}\pi\phi_0(s)\left(E + (sC + \Lambda P \circ (G - \overline{G}(s)))R\right) + sY(s), \qquad (12)$$

where $Y(s) =$

$$s\phi(s)\left((C + \Lambda P \circ (G - \overline{G}(s)))R\right)^2 - F(0)CR\left(E + (sC + \Lambda P \circ (G - \overline{G}(s)))\right).$$

Substituting $\phi(s)$ from (12) for $\phi(s)$ in (9) we can express $\phi_0(s)$ as follows

$$\phi_0(s) = \pi_0\frac{A(S)}{B_1(s) + B_2(s)}, \text{where} \qquad (13)$$

$$A(s) = \epsilon s - sY(s)\left(sC + \Lambda P \circ (G - \overline{G}(s))\right)e \qquad (14)$$

$$B_1(s) = \epsilon s + s^2 \pi \Lambda P \circ \frac{G - Ms - \overline{G}(s)}{s^2} \tag{15}$$

$$B_2(s) = s^2 \pi \left(C + \Lambda P \circ \frac{G - \overline{G}(s)}{s} \right) R \left(C + \Lambda P \circ \frac{G - \overline{G}(s)}{s} \right) e \tag{16}$$

Now, replacing s by $-\epsilon s$ in the equations (14), (15), (16), and letting $\epsilon \to 0$ and pointed out that

$$Y(-\epsilon s) \to 0, \frac{G - \overline{G}(-\epsilon s)}{-\epsilon s} \to M, \frac{\overline{G}(-\epsilon s) - G + M(-\epsilon s)}{(-\epsilon s)^2} \to V/2$$

we will have

$$A(-\epsilon s)/(-\epsilon^2 s) \to 1, B_1(-\epsilon s)/(-\epsilon^2 s) \to 1 + \frac{1}{2} \pi s \lambda_0 \Lambda^0 P \circ V e,$$

$$B_2(-\epsilon s)/(-\epsilon^2 s) \to -\pi \left(C + \lambda_0 \Lambda^0 P \circ M \right) R^0 \left(C + \lambda_0 \Lambda^0 P \circ M \right) e =$$
$$= \pi \left(C + \lambda_0 \Lambda^0 P \circ M \right) a^0,$$

where from (4) follows that a^0 is a solution to (2) and R^0 are defined for $\lambda = \lambda_0$. Combining these equations with (13) we get that there exists

$$\lim_{\epsilon \to 0} \phi_0(-\epsilon s) = \frac{\pi_0}{1 + Ns}, \text{and} N = \frac{1}{2} \pi \lambda_0 \Lambda^0 P \circ V e + \pi (C + \lambda_0 \Lambda^0 P \circ M) a^0.$$

Finally, from (12) we obtain

$$\lim_{\epsilon \to 0} \phi(-\epsilon s) = \lim_{\epsilon \to 0} \pi \phi_0(-\epsilon s)/\pi_0 = \pi/(1 + Ns)$$

which implies the results of the theorem. Thus, the theorem is proved.

4. Conclusion

We studied the fluid level in a fluid model introduced in [1]. We used the analytical approach introduced in [2] and derived that the scaled fluid level converges to an exponentially distributed random variable in heavy traffic. This limit random variable is independent of the state of the environment.

References

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