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Priority management in a semi-Markov queuing model

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Abstract. We study a single-channel queuing system with an arbitrary distribution of the duration of service requirements, on the input of which there are n Poisson processes. The requirements of the various processes come in different queues. The task is to determine the rule for selecting service requirements and to determine the optimal strategy for establishing dynamic priorities. We consider a case $n = 2$.

To this end, a controlled semi-Markov process is defined, on the trajectories of which a functional is constructed that determines the quality of management and takes into account the number of lost requirements, the number of serviced requirements, the time of the requirement stay in the system, and so on. An algorithm for determining the optimal strategy is formulated.

Keywords: optimal strategy, controlled semi-Markov process, dynamic priorities.

1. Introduction

At a research of queueing systems the structure of system often is considered invariable, and characteristics of the system are investigated at the fixed initial parameters and functions. It is important to note that in actual applications not only the research of stationary characteristics, but a possibility of structural change of the system and obtaining at the same time new results is of great interest.

When developing mathematical models the concept of strategy is entered [2], [3]. Strategy is meant as the rule of adoption of concrete decisions in particular instants. Further work of system depends on the made decisions. It is possible to change structure of system to increase in effectiveness of its work and calculation of characteristics at optimum functioning of the system (such as in papers [5], [4]).

2. Main section

The research of queueing system is investigated. The quantity of various arrival-flows is equal to n. Let's consider that each of flows represents a i-th type flow with the arriving queries of i-th type, i=*1,...,n*. The moments of numbers' arrival form each flow make the simplest flow of homogeneous moments with the parameter λ_i.

Queries of i-th type come to queue of i-th type , $i \in (1, 2, ..., n)$. The queue length for each flow is limited.

There is one channel in system. The channel can serve queries of any type. At the arbitrary moment here can be no more than one requirement on server. If there are no empty places in the queue, the query is lost. Durations of queries service are independent equally distributed random values. The cumulative distribution function of service duration changes depending on that what query type arrived for service. We assume that at the initial moment of functioning in system there are no requirements.

The $\boldsymbol{G}_i$ symbol means that the cumulative distribution function of service duration is arbitrary, but service duration depends on the type of the served requirement. That is if type of the served query i, then holding time is distributed under the law $G_i(x) = P\{\eta_i < x\}$. The maximal number of places in the queue of i-th type is equal $\boldsymbol{N}_i$.

In system discrete control is carried out by two factors - we choose: type of queue from which the following application arrives on the serving device; service duration for the query of i-th type.

The system can be considered as the controlled semi-Markov queueing system. At a research of the controlled semi-Markov queueing systems and construction of the controlled semi-Markov process describing evolution of the system we realize the following algorithm [1], [6].

The Markov moments in the case are the moments of the termination and the beginning of a query service.

Let's define a set of states of the queueing system at the decision making moments. Let's designate $\vec{l} = \{l_1, l_2, ..., l_n\}$, where l_k - the number of queries of k-th type turn at the time of decision-making, n- the quantity of queries types (amount of turns). The number of requirements in queue has terminating values, $l_k \in E_k = \{0, 1, 2, ..., N_k\}$, where N_k - the maximal number of places in the queue of k-th type. Thus $\vec{l} = (l_1, l_2, ..., l_n) \in E_1 \times ... \times E_n = E$.

The semi-Markov process $\xi(t)$at the moment t is defined as number of queries in each queue in the next, previous t, the Markov moment. In the accepted designations for a semi-Markov process it is possible to write down $\xi(t) \in E$.

The space of controls and the strategy of control is defined.

Each control consists of two components, and the set of possible controls depends on a state. Let's designate through $U(\vec{l})$ a set of possible decisions in a state $\vec{l}$. For a state $\vec{l}$ we will designate a vector $k^+(\vec{l}) = (n_1, n_2, ..., n_k)$,$0 \leq k \leq n$,$1 \leq n_1 < n_2 < ... < n_k \leq n$ for which components the ratio $l_{n_i} > 0$,$i = 1, 2, ..., k$ is carried out. In other words, for each state the set of turns in which there are requirements is defined. Let's designate through v type of the requirement or type of queue of which the following requirement for service gets out. Therefore $\nu \in k^+(\vec{l})$. If to designate through u duration of service, then it is apparent that this duration can accept any nonnegative values, that is $u \in [0, +\infty) = R^+$. Thus we have $(\nu, u) \in U(\vec{l}) = k^+(\vec{l}) \otimes R^+$.

We define Markov homogeneous randomized strategy, so is we will construct a probability measure on control set. For a discrete component v- the choice of queue type from which the application will arrive on service, we will put down

$$P\{v = k/\overrightarrow{l}\} = \begin{cases} 0, k \notin k^+(\overrightarrow{l}), \\ p_k, \end{cases} \qquad p_k \geq 0, \sum_{k \in k^+(\overrightarrow{l})} p_k = 1, \quad (1)$$

where through $P\{v = k/\overrightarrow{l}\}$ the probability to the query of k-th type for service is designated.

Components of managements dependent. Therefore we define distribution of the second component as the conditional probability

$$G(t, k, \overrightarrow{l}) = P\{u < t/\nu = k, \overrightarrow{l}\} \quad (2)$$

Elements of a semi-Markov matrix are defined as probabilities that some state will be the following state of a semi-Markovian process $\vec{l'} = (l'_1, l'_2, ..., l'_n) \in E$ and transition to this state will happen until t provided that process passed into a state $\vec{l} = (l_1, l_2, ..., l_n)$ and the decision $(\nu, u) = (k, \tau)$ is made. We will designate this probability $Q_{\vec{l}\vec{l'}}(t, k, \tau)$.

If the requirement of k-th of type begins to be served, duration of service is equal τ and process stayed in state $\vec{l} = (l_1, l_2, ..., l_n)$,$l_k \geq 1$, then in time τ this query will be served, and m_i of queries will come to turn of i-th type with probability

$$\pi(\lambda_i, \tau, m_i) = \frac{(\lambda_i \tau)^{m_i}}{(m_i)!} e^{-\lambda_i \tau} \quad (3)$$

Let's notice that at $i \neq k$, if $m_i > N_i - l_i$, then there are lost queries and their number is equal $m_i - N_i + l$, at $i = k$ number of the lost queries equals $m_k - N_k + l_k - 1$ as one query left to service.

We will give reasonings for a case with two types of applications, *n=2*.

The system state is described by (l_1, l_2), where l_1, l_2 - the number of queries in the queue of the 1st and 2nd type respectively, $0 \leq l_1 \leq N_1, 0 \leq l_2 \leq N_2$.

We give expressions for a *semi-Markov matrix*.

$$\begin{gathered} Q_{(0,0)(1,0)}(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\left(1 - e^{-(\lambda_1+\lambda_2)t}\right), \\ Q_{(0,0)(0,2)}(t) = \frac{\lambda_2}{\lambda_1+\lambda_2}\left(1 - e^{-(\lambda_1+\lambda_2)t}\right) \end{gathered} \quad (4)$$

These elements of a semi-Markov matrix are defined as probability that until t the application of i-th type *(i=1,2)* came to empty system of the first.

If being in state (l_1, l_2) the decision to send the query of the 1st type for service is made, that is $v = 1$ and service duration is set u, then the semi-Markov kernel will take a form $Q_{(l_1,l_2)(l_1',l_2')}(t, 1, u) =$

$$\begin{cases} 0, t \le u, \\ \frac{(\lambda_1 u)^{l_1'-l_1+1}}{(l_1'-l_1+1)!} e^{-\lambda_1 u} \frac{(\lambda_2 u)^{l_2'-l_2}}{(l_2'-l_2)!} e^{-\lambda_2 u}, l_1 - 1 \le l_1' < N_1, l_2 \le l_2' < N_2, t > u, \\ \frac{(\lambda_1 u)^{l_1'-l_1+1}}{(l_1'-l_1+1)!} e^{-\lambda_1 u} \sum\limits_{k=N_2-l_2}^{\infty} \frac{(\lambda_2 u)^k}{k!} e^{-\lambda_2 u}, l_1 - 1 \le l_1' < N_1, l_2' = N_2, t > u, \\ \frac{(\lambda_2 u)^{l_2'-l_2}}{(l_2'-l_2)!} e^{-\lambda_2 u} \sum\limits_{k=N_1-l_1+1}^{\infty} \frac{(\lambda_1 u)^k}{k!} e^{-\lambda_1 u}, l_1' = N_1, l_2 \le l_2' < N_2, t > u, \\ \sum\limits_{k=N_1-l_1+1}^{\infty} \frac{(\lambda_1 u)^k}{k!} e^{-\lambda_1 u} \sum\limits_{k=N_2-l_2}^{\infty} \frac{(\lambda_2 u)^k}{k!} e^{-\lambda_2 u}, l_1' = N_1, l_2' = N_2, t > u. \end{cases} \tag{5}$$

The semi-Markov kernel can be written out for each case by analogy with a formula (5).

The matrix of the transitional probabilities of the embedded Markov chain is bound to a semi-Markov matrix equality

$p_{(l_1,l_2)(l_1',l_2')}(\nu, u) = \lim\limits_{t\to\infty} Q_{(l_1,l_2)(l_1',l_2')}(t, \nu, u)$. Therefore, from equalities (4)-(5) easily we receive expressions for required probabilities.

Let's enter the cost characteristics defining the functional characterizing quality of functioning and control:

c_0^i- income from servicing one query of i-th type; c_1^i - payment per unit of time for servicing one query of i-th type on the service channel; c_2 - the expense per unit of time for the maintenance of the channel running idle; c_3^i - payment for one lost demand of i-th type; c_4^i - payment per unit of time for servicing one condition in the waiting room (in the queue) of i-th type.

These constants define the additive functional on trajectories of a semi-Markov process. We write out the *conditional expectations of the saved-up income* $R_{(l_1,l_2)(l_1',l_2')}(t, u, v)$ provided that process stays in state (l_1,l_2), through time of t will pass into a state (l_1',l_2') and the decision $\{u, v\}$ is made.

For any state (not $(0, 0)$) (l_1, l_2) and states $\left(l_1', l_2'\right), l_1 - 1 \le l_1' < N_1, l_2 - 1 \le l_2' < N_2$ we have

$$\begin{aligned} R_{(l_1,l_2)(l_1',l_2')}(u, u, 1) &= c_0^1 + c_1^1 u + c_4^1 \left(\frac{l_1'+l_1-1}{2}\right) u + c_4^2 \frac{l_2'+l_2}{2} u, \\ R_{(l_1,l_2)(l_1',l_2')}(u, u, 2) &= c_0^2 + c_1^2 u + c_4^2 \left(\frac{l_2'+l_2-1}{2}\right) u + c_4^1 \frac{l_1'+l_1}{2} u \end{aligned} \tag{6}$$

Conditional expectations of the saved-up income can be written out including for other states taking into account loss of queries.

Further we use the following facts:

1) For the income functional equality is fair [1], [6]:

$$S = \frac{\sum\limits_{(l_1,l_2)\in\tilde{E}} s_{(l_1,l_2)}\pi_{(l_1,l_2)}}{\sum\limits_{(l_1,l_2)\in\tilde{E}} m_{(l_1,l_2)}\pi_{(l_1,l_2)}} \qquad (7)$$

where $s_{(l_1,l_2)}$ are the conditional mathematical expectations of the accumulated income for the entire period of the process in the state $(l_1, l_2) \in E$,

$\pi_{(l_1,l_2)}$ – stationary probabilities, $m_{(l_1,l_2)}$ – are the conditional mathematical expectations of the time of the continuous stay of the process in the state $(l_1, l_2) \in E$;

2) The functional $S = S(\vec{G})$ is linear-fractional rather probability measures $\vec{G}$, defining Markov homogeneous randomized strategy [2] , [6] and optimum strategy can be looked for in a class of the determined strategy of control.

3. Conclusions

So we give the algorithm of the income functional construction and searching of optimum strategy for the system with two types of queries at control of service duration and the choice of query type for service.

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