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Controlled stochastic processes and control in queuing, reliability and safety

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Abstract. The functionals constructed on trajectories of the controlled semi-Markov process with a finite set of states are investigated. Theorems of functionals' structure (dependences on the probability measures defining the Markov homogeneous randomized strategy of control) and of structure of probability measures on which the extremum of these functionals is reached, are formulated. Examples are given.

Keywords: security, controlled semi-Markov process with disasters, homogeneous Markov randomized control strategy, reliability, failure-free operation, maintainability, optimal control.

1. Introduction

Jump-like processes (renewal processes, Markov processes, semi-Markov processes) a long time and widely are used for analyzing of queueing models and of models of reliability. Now these processes involved for description and analysis of security issues. Study of controlled random processes allows to put the task of optimization and to expand the area of application. When solving these task, there are two problems: how functionals, defining the quality of governance, depends on strategies of management and how to determine the optimal strategy of control. Addressing these issues is set out below.

2. Main section

The object of research is the homogeneous controlled semi-Markov process $X(t)$. The homogeneous controlled semi-Markov process

$$X(t) = (\xi(t), u(t))$$

is determined [1, 4] by a semi-Markov kernel $Q_{ij}(t,u),\ i,j \in E,\ t \geqslant 0,\ u \in U_i$, by a probability measures $\vec{G} = (G_i, i \in E)$ and by the initial distribution of the leading component $\xi(0)$, where E is the spaces of stats of leading component, $\xi(t) \in E$,

$(U_i, A_i), i \in E$ is the measurable sets of control parameters, $u(t) \in U_i$, if $\xi(t) = i$, which some σ-algebra A_i of their subsets.

We will define a functional $S_i(t)$ as the conditional expected value of complete effect to time $[0,t)$ if $\xi(0) = i \in E$.

Let be $\xi(0) = i$, $u(0) = u$, $\theta = t$ ($\xi(x) = i$, $0 \leq x < \theta = t$) and $\xi(t) = j$ — a next value of leading component. We will designate $R_{ij}(y,t,u)$ — as the conditional expected value of effect to time $[0,y)$ if $\xi(0) = i \in E$, $\xi(t) = j \in E$, $u(0) = u$, $\theta = t$. The functions $R_{ij}(y,t,u)$ are defined in area of $0 \leq y \leq t < \infty$, $i,j \in E$, $u \in U_i$ We will notice that in the model of function $R_{ij}(y,t,u)$ is determined as the conditional expected values of the accumulated effect, as in certain problems the controlled semi-Markov process is often examined as the inlaid process in more thin casual process that changes the states between Markov moments. Let $R_{ij}(t,t,u) = R_{ij}(t,u)$. We assume that for disjoint time intervals accumulated effect (income) summed up. We will define a functional accumulation $S_i(t)$ as the conditional expected value of complete effect to time $[0,t)$ if $\xi(0) = i \in E$. The object of study is the limit of the ratio $\frac{S_i(t)}{t}$ (if that limit exists) for $t \to \infty$

Define the functionality of the other. We divide the set E into two disjoint subsets E_0 and $E \setminus E_0$. Then each trajectory of semi-Markov process we put into correspondence the time τ of the first occurrence of the trajectory in subset $E \setminus E_0$ and shall study the functional $S_1^{(i)}$, $i \in E_0$ - the conditional mathematical expectation the accumulated income before the time τ, if $\xi(0) = i$.

The following two problems arise for the functional defined on a trajectories controlled semi-Markov process $X(t) = (\xi(t), u(t))$. First we have to clarify how the functional $S_i = \lim_{t\to\infty} \frac{S_i(t)}{t}$ and the functional $S_1^{(i)}$ depend on the probability measures $\vec{G} = (G_i, i \in E)$ which determine a homogeneous Markov randomized control strategy. Secondly, we want to find the extremum of these functionals and determine the structure of the distributions $\vec{G} = (G_i, i \in E)$ on which the extremum is reached.

Now we will set forth a main theorem.

Let Ω is a set of acceptable strategy, E_0 is a subset of the unimportant states, $E_k, k = 1, 2, ..., K, K < \infty$ are subsets of the substantial states.

THEOREM 1. [5]

Suppose:

1. The set E is finite;

2. For any acceptable strategy $\vec{G} = (G_i, i \in E) \in \Omega$ the process $X(t) = (\xi(t), u(t))$ is regular;

3. Conditions

$$m_i = \sum_{j\in E} \int_{U_i} \int_0^\infty z dQ_{ij}(z,u) G_i(du) < \infty,$$
$$\sum_{j\in E} \int_{U_i} \int_0^\infty z^2 dQ_{ij}(z,u) G_i(du) < \infty,$$
$$\int_0^\infty \int_{U_i} \int_0^z x d_x R_{ij}(x,z,u)] dQ_{ij}(z,u) G_i(du) < \infty$$

are true;

4. For any $k = 1, 2, ..., K$ there exists $i_k \to E_k$ for which

$$s_{i_k} = \sum_{j \in E} \int_{U_i} \int_0^{\infty} R_{i_k j}(z, u) dQ_{i_k j}(z, u) G_{i_k}(du) \neq 0.$$

Then

$$S_i(t) = S_i t + O(1), \qquad t \to \infty, \tag{1}$$

and S_i is fractional - linear functional with respect to the probability measure, defining homogeneous Markov randomized control strategy, and

$$S^{(k)} = S_i(G^{(k)}) = \frac{I_1^{(k)}(G_i,\, i \in E_k)}{I_2^{(k)}(G_i,\, i \in E_k)} \qquad i \in E_k, \quad k = 1, 2, \ldots, K,$$

$$S_i = S_i(\vec{G}) = \sum_{k=1}^{K} \gamma_i^{(k)} S^{(k)} \qquad i \in E_0$$

$$\gamma_i^{(k)} = \frac{I_{ik}^{(0)}(G_n,\, n \in E_0)}{I^{(0)}(G_n,\, n \in E_0)}, \qquad i \in E_0,$$

Functionals $S_1^{(i)}, i \in E_0$ have a similar structure.

THEOREM 2.

If a semi-Markov process has a finite number of states $E = (1, 2, ..., N)$, $N < \infty$, a set $E_0 = (1, 2, ..., n)$, is a subset of non-recurrent states, $E_1 = (n+1, n+2, ..., N)$, $0 < n < N < \infty$, is a set of absorbing states of embedded chain of Markov , then the functionals $S_1^{(i)}, i \in E_0$ are fractional - linear functionals with respect to the distribution functions $\vec{G} = (G_i, i \in E_0) \in \Omega$.

Problem fractional - linear programming can be reduced to problem linear programming.

Denote fractional - linear functional

$$I(G_1, G_2, ..., G_N) = \frac{I_1(G_1, G_2, ..., G_N)}{I_2(G_1, G_2, ..., G_N)} = \frac{I_1(\vec{G})}{I_2(\vec{G})} \tag{2}$$

where for $i = 1, 2$

$$I_i(\vec{G}) = \int_{U_1} \int_{U_2} ... \int_{U_N} A_i(u_1, u_2, ..., u_N) G_1(du_1) G(du_2) ... G(du_N),$$

Ω_1 is a set of acceptable strategy $\vec{G}$ such that $I_2(\vec{G}) > 0$, Ω_2 is a set of acceptable strategy $\vec{G}$ such that $I_2(\vec{G}) < 0$.

LEMMA. [2, 3] If the maximum of the linear-fractional functional (2) over the set of acceptable strategy Ω exists, $\max_{\vec{G}\in\Omega} I(\vec{G}) = c$, then

$$\{\vec{F} : I(\vec{F}) = \max_{\vec{G}\in\Omega} I(\vec{G}) = c\} =$$

$$\{\vec{F} : I_1(\vec{F}) - cI_2(\vec{F}) = \max_{\vec{G}\in\Omega}[I_1(\vec{G}) - cI_2(\vec{G})] = 0\} = \bigcup_{i=1}^{2} H_i,$$

$$H_i = \{\vec{F} : I_1(\vec{F}) - cI_2(\vec{F}) = \max_{\vec{G}\in\Omega_i}[I_1(\vec{G}) - cI_2(\vec{G})] = 0\}.$$

Next, we analyze the structure of strategies on which an extremum of a fractional-linear functional is attained.

Let be true $\{u_i\} \in A_i, u_i \in U_i, i \in E$ and $\vec{G}^* = (G_i^*, i \in E)$,

$$G_i^*(\{u_i\}) = 1, G_i^*(B) = 0, B \neq \{u_i\}. \tag{3}$$

We denote by Ω^* the set of strategies for which the relations (3).

THEOREM 3. [6] If the maximum of the linear-fractional functional (2) exists and $\Omega^* \in \Omega$, then

$$\max_{\vec{G}\in\Omega} I(\vec{G}) = \max_{\vec{G}\in\Omega^*} I(\vec{G}) = \max_{u_i\in U_i, i\in E} \frac{A_1(u_i, i \in E)}{A_2(u_i, i \in E)}$$

that is, the optimal distribution can be found in class Ω^* of degenerate distributions.

The examples.

By this model we study the controlled semi-Markov queues, models of maintenance and models of safety.

1. *Queueing system* [4,7]. We define the optimal duration of the service (the dependence on the state of the system).

2. *Queueing system* [7]. Supposed that recurrent flow arrive in system which have n servers and N waiting places, the distribution function $G(t)$ interval between the moments of the arrival depend on state of system. We shell control by the interval between the moments of the arrival and select this interval (the distribution function) depending on number of demands at queuing system.

3. *Model maintenance of redundancy system* [6]. We study a system of two equivalent components (a basis element and a back-up element, which do not come out of action, if his is found at the place of back-up element). The damaged component is repaired by one worker, we shall call this repair as repair after failure. In model it is possible also two kinds of preventive repair: scheduled repair and non scheduled repair. The all kinds of repair make element new. Built a semi-Markov process which changes its state at the start of the repairs of one of the elements.

4. *Model safety* [8]. Consider the model of functioning of technical systems that protect information. Since the system is going to fail, and the attacker can obtain during these periods, information, solves the problem of choosing the optimal period for preventive maintenance, ensuring the maximum expectation time to the loss of information.

3. Conclusions

The theorems formulated above allow us to construct an algorithm for computing the optimal control strategy for the controlled queuing models, reliability and security, in the case where the operation of such systems are described by semi-Markov process with finite set of states.

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