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Analysis of random neural networks with an infinite number of cells

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Abstract. In this paper we consider dynamics of complex systems using random neural networks with an infinite number of cells. The Cauchy problem for singular perturbed infinite order systems of stochastic differential equations which describes the random neural network with infinite number of cells is studied.

Keywords: analytical methods in probability theory, asymptotic methods, neural network; dynamics of complex systems.

1. Introduction

The recent research of random neural networks with an infinite number of cells deal with the problem of the solutions analysis of certain infinite systems of ordinary differential equations. A model for a large network of "neurons" with a graded response (or sigmoid input-output relation) was studied [4]. The idea was used in biological systems was given added credence by the continued presence of such properties for more nearly biological "neurons". In the paper [1] was given existence and uniqueness results for the equations describing the dynamics of some neural networks for which there were infinitely many cells. Such system was considered and neural nets which were modeled were described by the singular perturbed infinite system of ordinary differential equations. Bruce D. Calvert and Armen H. Zemanian [2] investigated a nonlinear infinite resistive network, an operating point could be determined by approximating the network by finite networks obtained by shorting together various infinite sets of nodes, and then taking a limit of the nodal potential functions of the finite networks. Claudio Turchetti and other author [5], [6] provided applications of stochastic neural networks.

In paper [7], we considered neural networks with an infinite number of cells. The Cauchy problem for infinite order systems of differential equations is considered. The theorems of existence and uniqueness of solution is proved.

In this paper we propose method of analysis random neural networks with an infinite number of cells. Cauchy problem is studied for singular perturbed infinite order systems of differential equations with random coefficients, which describes the stochastic process in neural network with infinite number of cells.

2. Large scale random neural networks model

In 1984 Hopfield investigated a neural network which was described using system of ordinary differential equations [4]

$$C_i \frac{du_i}{dt} = \sum_{j=1}^{N} T_{ij} g(u_j(t)) - \frac{u_i(t)}{R_i} + I_i,\ i,j = 1, \ldots, N,\ t \geq 0, \qquad (1)$$

where $u_i(t) \in \mathbf{R}$ is a monotone-increasing function of the instantaneous input to neuron i, $C_i > 0$ is a capacitance of the cell membrane, $R_i > 0$ is a transmembrane resistance, $I_i > 0$ is a fixed input current to neuron i and $t \in \mathcal{T}$ ($\mathcal{T} \in \mathbf{R}_+$) is a time parameter. The matrix element $T_{ij} \in \mathbf{R}$ can be assumed as a description of the synaptic interconnection strength from neuron j to neuron i and T_{ij}^{-1} is a finite impedance between the output V_j and the cell body of cell i, where $u_i = g_i^{-1}(V_i)$ and $g(u_i(t)) \in [-1;1]$ is an increasing continuous function from $\mathbf{R}$ to $[-1;1]$, perhaps $g(u_i(t)) = tanh(u_i(t))$.

We can rewrite the system (1) in the form

$$\begin{aligned} &\frac{du_i}{dt} = f_i(\mathbf{u}, g, t) + a_i u_i(t) + b_i,\ i,j = 1, \ldots, N,\ t \geq 0, \\ &f_i(\mathbf{u}, g, t) = \sum_{j=1}^{N} \frac{T_{ij}}{C_i} g(u_j(t)),\ a_i = -\frac{1}{C_i R_i},\ b_i = \frac{I_i}{C_i}, \end{aligned} \qquad (2)$$

where $\mathbf{u} \in \mathbf{R}^N$ $(u_1, u_2, \ldots, u_N)$, $\mathbf{f} \in \mathbf{R}^N$ $(f_1, f_2, \ldots, f_N)$ are N-dimensional vector functions and $\mathbf{a} \in \mathbf{R}^N$ $(a_1, a_2, \ldots, a_N)$, $\mathbf{b} \in \mathbf{R}^N$ $(b_1, b_2, \ldots, b_N)$ are N- dimensional vectors.

We can generalize the system (2) assuming $g(u_j(t))$ is a random function

$$\begin{aligned} &\frac{du_i}{dt} = f_i^N(\mathbf{u}, g, t; \omega) + a_i u_i(t) + b_i,\ i,j = 1, \ldots, N,\ t \geq 0, \\ &f_i^N(\mathbf{u}, g, t; \omega) = \sum_{j=1}^{N} \frac{T_{ij}}{C_i} g(u_j(t), \omega), \end{aligned} \qquad (3)$$

where $g(u_j(t);\omega) \in [-1;1]$ is an increasing random function ($(t;\omega) \in \mathcal{T} \times \Omega$, $(\Omega; \boldsymbol{F}; \mathcal{P})$ is an abstract probability space).

For describing rapid changes of processes in some element of the neural network we can use a small parameter $\mu > 0$ that bring a singular perturbation to the system (3)

$$\begin{cases} \frac{du_i}{dt} = f_i^N(\mathbf{u}, g, t; \omega) + a_i u_i(t) + b_i,\ i = 1, \ldots, n, \\ \mu^{s_i} \frac{du_i}{dt} = f_i^N(\mathbf{u}, g, t; \omega) + a_i u_i(t) + b_i,\ i = n+1, \ldots, N,\ n < N, \end{cases} \qquad (4)$$

where $s_i \in \mathbf{N}$ $(0 < s_{n+1} \leq s_{n+2} \ldots \leq s_N)$ is a finite sequence of natural numbers and $i \in (n+1, \ldots, N)$ is the numbers of neural cells in which the speed of the processes faster than in neural cells with numbers $i \in (1, \ldots, n)$.

For the system (4) we can use extended variables $\tau_i = t/\mu^{s_i}$ where $s_i = 0$ when $i \in (1, \ldots, n)$ or $\mathbf{s} \in \mathbf{R}^N$ $(0, \ldots, 0, s_{n+1}, \ldots, s_N)$ and rewrite it in the form

$$\frac{du_i}{d\tau_i} = f_i^N(\mathbf{u}, g, \tau_i; \omega) + a_i u_i(\tau_i) + b_i,\ i = 1, \ldots, N, \tag{5}$$

where $(\tau_i; \omega) \in \mathcal{T}_i \times \Omega$ $((\Omega; \boldsymbol{F}; \mathcal{P})$ is an abstract probability space).

For system (5) we can formulate the following Cauchy problem

$$\begin{cases} \frac{du_i}{d\tau_i} = f_i^N(\mathbf{u}, g, \tau_i; \omega) + a_i u_i(\tau_i) + b_i, \\ u_i(0) = \bar{u}_i^0,\ i = 1, 2, \ldots, N, \end{cases} \tag{6}$$

where a finite numerical sequence $(\bar{u}_1^0, \bar{u}_2^0, \ldots, \bar{u}_N^0)$ or $\bar{\mathbf{u}}^{\mathbf{0}} \in \mathbf{R}^N$ is the initial conditions of the problem (6). The sequence $\bar{\mathbf{u}}^{\mathbf{0}}$ determines the initial state of the neural network.

We can consider the neural network with infinite number of cells and we can formulate the following Cauchy problem in case $N \to \infty$ in the form

$$\begin{cases} \frac{du_i}{d\tau_i} = f_i(\mathbf{u}, g, \tau_i; \omega) + a_i u_i(\tau_i) + b_i, \\ u_i(0) = \bar{u}_i^0,\ i = 1, 2, \ldots, \end{cases} \tag{7}$$

where

$$f_i(\mathbf{u}, g, \tau_i; \omega) = \sum_{j=1}^{\infty} \frac{T_{ij}}{C_i} g(u_j(\tau_i), \omega),\ i = 1, 2, \ldots,$$

and $\mathbf{u} \in \mathbf{R}^\infty$ $(u_1, u_2, \ldots)$, $\mathbf{f} \in \mathbf{R}^\infty$ $(f_1, f_2, \ldots)$ are infinite dimensional vector functions and $\mathbf{a} \in \mathbf{R}^\infty$ $(a_1, a_2, \ldots)$, $\mathbf{b} \in \mathbf{R}^\infty$ $(b_1, b_2, \ldots)$ are infinite dimensional vectors. The initial conditions $(\bar{u}_1^0, \bar{u}_2^0, \ldots)$ of the problem (7) or $\bar{\mathbf{u}}^{\mathbf{0}} \in \mathbf{R}^\infty$ determines the initial state of the neural network.

3. Random neural networks with an infinite number of cells modeling

We can consider the following Cauchy problem for infinite order systems of differential equations with a random coefficient

$$\begin{cases} \frac{du_i}{d\tau_i} = \epsilon_{\tau_i}\, u_{i+1}(\tau_i), \\ u_i(0) = \bar{u}_i^0,\ i = 1, 2, \ldots, \end{cases} \tag{8}$$

where $\epsilon_{\tau_i} \cong N(r, \sigma^2)$ are independent normally distributed random variables. For this infinite order systems of differential equations we can study a similar problem without a stochastic variable such form

$$\begin{cases} \frac{du_i}{d\tau_i} = u_{i+1}(\tau_i), \\ u_i(0) = \bar{u}_i^0,\ i = 1, 2, \ldots, \end{cases}$$

Therefore, at least one solution of this system of equations passes through every point $(0, 0, \ldots, \bar{u}_1^0, \bar{u}_2^0, \ldots)$. It is easy to see that the series

$$u_i(\tau_i) = \bar{u}_i^0 + \tau_i \bar{u}_{i+1}^0 + \frac{\tau_i^2}{2} \bar{u}_{i+2}^0 + \ldots,\ i = 1, 2, \ldots,$$

define the required solution. Now we can obtain the exact solution of the problem (8)

$$u_i(\tau_i) = \bar{u}_i^0 \int_0^\infty u_i\left(\tau_i \zeta\right) f(\zeta)\, d\zeta,\ i = 1, 2, \ldots,$$

where $f(\zeta)$ is the probability density function. If ϵ_{τ_i} is a Gaussian random variable with mean r and variance σ, then the integral can be evaluated exactly if the initial condition of Chauchy problem we get as $\bar{u}_i^0 = 1$, $i = 1, 2, \ldots,$

$$u_i(\tau_i) = \int_0^\infty \exp\left(\tau_i \zeta\right) f(\zeta)\, d\zeta,\ i = 1, 2, \ldots,$$

$$u_i(\tau_i) = \exp\left(r\tau_i + \frac{\sigma^2 \tau_i^2}{2}\right),\ i = 1, 2, \ldots$$

or turning to the usual variables we get solutions Cauchy problem

$$u_i(t) = \exp\left(r\mu^{-s_i} t + \frac{\sigma^2 \mu^{-2s_i} t^2}{2}\right),\ i = 1, 2, \ldots$$

We can solve infinite order systems of differential equations with random coefficients using wide class of continuous transfer functions.

4. Conclusions

In this paper we propose method of analysis random neural networks with an infinite number of cells. Cauchy problem for singular perturbed infinite order systems of differential equations with random coefficients, which describes the stochastic process in neural network with infinite number of cells, is studied. For this Cauchy problem consider the question of the existence of solution and obtain some exact solutions.

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References

1. *Calvert B. D.* Neural networks with an infinite number of cells // Journal of Differential Equations. — 2002. — Vol. 186, no. 1. — P. 31–51.
2. *Calvert B. D., Zemanian A.H.* Operating points in infinite nonlinear networks approximated by finite networks // Trans. Amer. Math. Soc. — 2000. — Vol. 352, no 2. — P. 753–780.
3. *Henry D.* Geometric theory of semilinear parabolic equations. Lecture Notes in Mathematics. — Springer-Verlag, Berlin, 1981.
4. *Hopfield J. J.* Neurons with graded response have collective computational properties like those of two state neurons // Proc. Natl. Acad. Sci. USA. — 1984. — Vol. 81. — P. 3088 – 3092.
5. *Turchetti C.* Stochastic Models of Neural Networks, Frontiers in artificial intelligence and applications: Knowledge-based intelligent engineering systems. — IOS Press, 2004. — Vol. 102.
6. *Turchetti C., Crippa P., Pirani M., Biagetti G.* Representation of Nonlinear Random Transformations by Non-Gaussian Stochastic Neural Networks // IEEE transactions on neural networks / a publication of the IEEE Neural Networks Council. — 2008.— Vol. 19. — P. 1033—1060.
7. *Vasilyev S. A., Kanzitdinov S. K.* Model of neural networks with an infinite number of cells and small parameter // International scientific journal "Modern Information Technologies and IT-Education". — 2016. — Vol. 12, no. 2. — P. 15—20.