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Unpublished Manuscripts by G.W. Leibniz Associated with Nondecimal Systems

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Abstract. The paper presents unpublished manuscripts from Leibniz Archive in Hannover on binary fractions and hexadecimal numbers. The research of Leibniz's correspondence and manuscripts shows that he knew the principles of binary fraction representation, could operate them, and was the first who proposed to apply the binary system for calculating transcendental numbers. At the same time to calculate binary fractions he used either division or the method of undetermined digits. Evidently he did not know any efficient algorithm of conversion into binary fractions. Also Leibniz understood the connection between hexadecimal and binary systems, knew the algorithm for converting integers to hexadecimal system, introduced several methods to present hexadecimal digits and suggested the practical use of the hexadecimal system instead of the binary system that nowadays came true in computer science and mathematics.

Keywords: history of mathematics, G. W. Leibniz, unpublished manuscripts, binary system, hexadecimal system.

1. Introduction

The history of the binary and hexadecimal number systems is interesting due to their wide use in theoretical computer science and computer technology. Nondecimal systems have been the subject of research and tool for problem solving in the works of many mathematicians, although not all the results had been published.

The German mathematician and philosopher Gottfried Wilhelm Leibniz (1646—1716) is considered the founder of binary arithmetic, despite the fact that the English mathematician Thomas Harriot (1560—1621) had used the binary system in his manuscripts long before Leibniz [1].

First time Leibniz introduced the binary arithmetic in his manuscript "De Progressione Dyadica" (About binary progression) dated March 15, 1679 [2]. Leibniz presents the binary representations of numbers from 1 to 100, proposes an algorithm of converting integers into the binary system by means of multiple division by 2 calculating binary digits as remainders, presents examples of addition, subtraction, multiplication and division in the binary system.

Leibniz's article in 1703 [3] was the first publication on the binary system. However in this article Leibniz doesn't present algorithms for converting integers into the binary system and back. Leibniz points out the conformities in sequences of binary digits in the numbers from an arithmetic progression. Binary fractions are not mentioned in the article.

G. Zacher published in 1973 the fullest study on Leibniz's writings on the binary system [4]. This edition contains a list and a detailed review of Leibniz's works as well as 28 original Leibniz texts including 23 letters from Leibniz's correspondence and 5 separate manuscripts. In the published texts binary fractions are mentioned only in some letters at the stage of definition, first of all calculating the number π that Leibniz called Ludolph's number after Ludolph van Ceulen who had calculated its value with 35 decimal digits. In 1703—1705 Leibniz discussed the use of binary arithmetic to calculate Ludolph's number in his letters with Jacob Hermann, Joachim Bouvet, César Caze, and Jakob Bernoulli [5].

To fully appreciate the results of Leibniz's research a study of his manuscripts from the Leibniz Archive in Hannover was performed. More than 1,500 sheets of manuscripts have been reviewed. Except 78 notes on 105 pages included in the archive folder LH XXXV, 3b, 5 ("Different manuscripts on the binary arithmetic") we have found 24 documents from the other sections related to nondecimal systems. 14 unpublished notes on the binary fractions and 3 notes related to the hexadecimal system are the most important documents for our study. Their content is presented in details below.

2. Binary fractions in unpublished manuscripts by G. W. Leibniz

First time Leibniz used binary fractions in unpublished second part of the above mentioned "De Progressione Dyadica" dated March 15, 1679 (LH XXXV, 3b, 2, p.3r-5). He writes the binary expressions for the fractions from $\frac{1}{2}$ to $\frac{1}{9}$, but doesn't separate the fractional part. To calculate a binary fraction Leibniz divides the numerator by the denominator using "the galley method". He stops calculating when he finds the period of binary fraction. Leibniz verifies his calculation, for example, he adds the fractional part of $\frac{4}{5}$ obtained by 2 digit left shift from $\frac{1}{5}$ and $\frac{1}{5}$, as result he obtains the fraction corresponding to integer 1. Similar calculations can be found in other Leibniz notes.

Also Leibniz suggests an interesting method for determining binary digits with the use of undetermined coefficients (LH XXXV, 3b, 5, p.11). He denotes the binary digits in the period of $\frac{1}{5}$ by the Latin letters *lmnp*, then adds $\frac{1}{5}$ to $\frac{4}{5}$ (shifted to 2 digits left) and using the known result he calculates the values of the digits

$$\begin{array}{cccccccccccccc}
 & & l & m & n & p & l & m & n & p & l & m & n & p \\
l & m & n & p & l & m & n & p & l & m & n & p & & \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array},$$

where $p = n = 1, m = l = 0$.

Similarly Leibniz finds the period 001 for $\frac{1}{7}$ adding 3 values: the fraction without shift, with 1 digit shift and with 2 digits shift (using the equality $\frac{1}{7} + \frac{2}{7} + \frac{4}{7} = 1$). He uses the method to calculate the period 000111 for $\frac{1}{9}$.

On following page (LH XXXV, 3b, 5, p.12) Leibniz tries to apply his method for the binary representation of $\frac{1}{11}$. Denoting sequential digits of the fraction $\frac{1}{11}$ as $\gamma\beta\alpha zyxwutvqpml$, Leibniz expresses the sum of 3 expressions: the fraction, the fraction with 1 digit left shift, and the fraction with 3 digit left shift. In fact he expressed the sum $\frac{1}{11} + \frac{2}{11} + \frac{8}{11} = 1 = 0,1111111...$ This gave him the possibility to calculate the digits l=1, m=0, n=1, p=1, q=1, r=0, s=1, t=0, u=1, w=0, x=0, y=0. At this point he stops the calculation, apparently having found an error (in fact u=0).

In other note (LH XXXV, 3b, 5, p. 89) Leibniz tries to obtain binary expressions for fractions through multiplication of the already obtained ones. For example, he tries to calculate the expression for $\frac{1}{15}$ as $\frac{1}{3} \cdot \frac{1}{5}$. Obtaining the erroneous intermediate outcome 0,0000111100001111 he converts it into the known right answer 0,000100010001 adding some periodic fraction. Then he tries to obtain a binary representation for $\frac{1}{9}$ as $\frac{1}{3} \cdot \frac{1}{3}$, but only the first 9 digits after the point are correct.

In "Pro fractionibus dyadica exprimendi" (Binary fraction expressions, LH XXXV, 3b, 5 p. 2-3) dated December 20, 1699, Leibniz begins with the binary expression for $\frac{1}{3}$, then introduces the notation *(b)* (b in brackets) for the number he calls "anti-b", i.e., if $b = 0$, then $(b) = 1$ and vice versa. Further on, he deduces some formulae for *(b),* trying to use them for calculating products of binary fractions in a generalized form.

The note "Tentata expressio circuli per progressionem dyadicam" (The attempt to express the circle through binary progression, LH XXXV 12, 2, p.97) describes the initial steps of conversion $\frac{\pi}{4}$ into the binary system. Leibniz tries to compare the sum of several terms of the progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}...$ with the sum of the series for $\frac{\pi}{4}$. So he deduces the following inequalities $\frac{1}{2} < 1 - \frac{1}{3}$, $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} < \frac{1}{2} + \frac{1}{4} < 1 - \frac{1}{3} + \frac{1}{5}$. In the next step Leibniz makes a mistake and arrives at the wrong conclusion $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} < \frac{1}{2} + \frac{1}{4}$, so that the digit 0 must be placed in second position after the point. Of course his further attempts to evaluate the sum of the series were unsuccesfull. In another note without a title (LH XXXV, 13, 3, p.33-34) Leibniz expressed $\pi/4$ as a sum of a double row, in which the denominators are powers of two, but no simplification has reached in the summation.

3. The hexadecimal system in unpublished manuscripts by G. W. Leibniz

It's worth remarking that Leibniz was working on the hexadecimal system at the same time as on the binary in "Sedecimal progression" dated 1679 (Hexadecimal progression, LH XXXV, 13, 3 p. 23). For the greater part the note deals with the rules for converting decimal numbers to the hexadecimal number system. Leibniz presents the algorithm of convertion using the example of 1679 (the year of writing the note).

The algorithm is based on sequential division by the powers of 16: 4096, 256, 16, 1. First the smallest power 256 is chosen which doesn't exceed the initial number. The division of 1679 by 256 gives 6 which is the first digit of the number. Leibniz notices that hexadecimal presentation of 1679 has 6 in hundreds. The remainder 143 is divided by 16, given 8 and a new remainder 15. Thus Leibniz obtained three digits 6, 8, 15.

The digits between 10 and 15 can be represented in different ways. At the top of this page Leibniz uses sequential Latin letters m, n, p, q, r, s. He skipped the letter o, probably not to be mistaken for zero. Next Leibniz represents the digits between 10 and 15 through the initial letters at the time used for musical notes (*Ut*, *Re*, *Mi*, *Fa*, *Sol*, *La*). Leibniz also introduced the names for hexadecimal numerals from 1 to 30 combining German words and affixations with Latin names for musical notes. For example, the decimal number 42 in hexadecimal system is presented as $2u$ and pronounced *utzwanzig*. On the reverse page Leibniz converts the powers of 10 through 100000 into the hexadecimal number system. He obtains $10 = u$, $100 = 64$, $1000 = 3s8$, $10000 = 2710$, $100000 = 186u0$. On the same page he checks the result, multiplying 100 by 10 in hexadecimal system (i.e. 64 by u).

In the other notes (LH XXXV, 3b,17 p. 4r and LH XXXV, 3b, 5 p. 77) Leibniz introduces special signs for hexadecimal digits with values from 0 to 15 based on their binary representation. So the digit with value 11 has a binary presentation 1011 and is shown as the 4-component sign written in a column: dash, dot, dash and one more dash. Leibniz uses on more way for writing digits in a line. He writes a digit as the union of several arcs. An arc is convex upwards if it correspond to binary 1, an arc is convex downwards if it correspond to 0.

Here Leibniz also points out that the binary numeration is theoretical but the hexadecimal numeration is more practical. Note that Leibniz's prediction that the practical applications of the hexadecimal and the binary systems are possible in the search for regularities in the sequences of digits of transcendental numbers is a genius because in 1995 the Bailey-Borwein-Plouffe (BBP) formula was discovered [6], which allows you to calculate any number of the number π in hexadecimal notation without having to calculate the previous digits:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right).$$

Thus, the hexadecimal system has advantages over the decimal system just to calculate the number π. Similar formulas exist for some constants, for example, $\ln 2$.

4. Conclusions

The study of Leibniz's correspondence and manuscripts shows that he knew the principles of binary fraction representation, could operate them, and was the first who proposed to apply the binary system for calculating transcendental numbers. At the same time Leibniz did not know any efficient algorithm of conversion into binary fractions. So analyzing Leibniz's manuscripts we can see his understanding of the connection between hexadecimal and binary systems. He also knew the algorithm for converting integers to hexadecimal system, introduced several methods to present hexadecimal digits and suggested the practical use of hexadecimal system instead of binary system that nowadays came true in computer science and mathematics.

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References

1. *Shirley J. W.* Binary numeration before Leibniz // American Journal of Physics. — 1951. — Vol.19. — P. 452-454.
2. *Greve J., Gumin H., Hochstetter E.* Herrn von Leibniz' Rechnung mit Null und Eins — Berlin, München: SiemensAG, 1966. — P. 42–47.
3. *Leibniz G. W.* Explication de l'arithmétique binaire, qui se sert des seuls caractères 0 et 1 avec des remarques sur son utilité, et sur ce qu'elle donne le sens des anciennes figures Chinoies de Fohy // Histoire de l'Académie Royale des Sciences anné MDCCIII. Avec les mémoires de mathématiques. — 1705. — P. 85–89.
4. *Zacher H. J.* Die Hauptschriften zur Dyadik von G. W. Leibniz: Ein Beitr. zur Geschichte des binären Zahlensystems — Frankfurt a. M., Klostermann, 1973. — 384 p.
5. *Shukhman E. V.* About non-decimal representations of fractional numbers in works by mathematicians in XVII-XVIII centuries // Istoriya nauki i tekhniki. — 2013. — No. 1. — P. 3–16.
6. *Bailey D. H., Borwein P. B., Plouffe S.* On the Rapid Computation of Various Polylogarithmic Constants // Mathematics of Computation. — 1997. — Vol. 218. — No. 66. — P. 903—913.