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The idea of order in geometry, algebra, combinatorics in the 17th-19th centuries

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Abstract. Leibniz formulated the idea of constructing a special geometry - geometria situs, in which the basic relation is mutual arrangement of figures. He called the corresponding calculus an analysis situs. In the 19th century, Leibniz's ideas about the analysis situs were developed and implemented in combinatorics, projective geometry, and algebra. In combinatorics and projective geometry configurations have been actively investigated, substitutions have been analysed in algebra. Poincare created a new discipline, which he called first "Analysis Situs", and then "Topology".

Keywords: analysis situs, order, Leibniz, combinatorics, geometria situs, Tactic, tactical configuration, Sylvester, graph.

1. Introduction

Leibniz's idea of geometria situs stimulated research in combinatorics, projective geometry, and the theory of finite groups. Sylvester J.J. proposed to create a section of mathematics that studies the arrangement of elements, and called it Tactic. He referred to it combinatorics, number theory and algebra. The idea of Leibniz contributed to the emergence of topology, the graph theory; integration of sciences that study discrete structures; development of modern discrete mathematics.

2. Leibniz on analysis situs

Leibniz's idea of analysis situs arose from his thinking about universal science based on a universal language. Rene Descartes formulated the idea of universal science in "Discours de la methode. Pour bien conduire sa raison, et chercher la verite dans les sciences", 1637. One of the appendices of this work was "Geometry" which contained general rules of a scientific method. He proposed to reduce all mathematics to algebra. Descartes desire for mathematization of natural science, for the creation of universal science, was supported by Leibniz. Leibniz's idea of analysis situs refers to a new understanding of "geometric algebra". It played an exceptional role in the development of geometry and mathematics in general. In a letter to Ch. Huygens dated September 8, 1679, Leibniz wrote: "... I am no longer content with algebra, insofar as it gives neither the shortest nor the most elegant constructions in geometry. That is why... I think we still need another, properly geometrical linear analysis that will directly express for us situation, just as algebra expresses magnitude. I believe I have a method of doing this, and that we can represent figures and even machines and movements with characters, just as algebra represents numbers or magnitudes. I am sending you an essay that seems to me important ..." (Leibnizens mathematische Schriften, Bd.2, S. 17-25).

3. From analysis situs to geometry on a chessboard

In a letter to P. R. de Montmort written on 17th January 1718 G. W. Leibnitz wrote (in French): "The game called Solitaire pleases me much. I take it in reverse order. That is to say, instead of making a configuration according to the rules of the game, which is to jump to an empty place and remove the piece over which one has jumped, I thought it was better to reconstruct what had been demolished by filling an empty hole over which one has leaped". Following Leibniz, L. Euler (1758) and Ch. A. Vandermonde (1771) were engaged in geometry on the chessboard. Euler solved the knight's tour problem – the problem of finding a sequence of moves of a knight on a chessboard such that the knight visits every square only once. Euler considered square, cross-shaped and rectangular boards. Vandermonde generalized the knight's tour problem to the three-dimensional case.

4. Development of the ideas of Leibniz in the XIX century geometry

The geometric ideas of Leibniz were further developed in the 19th century in the construction of regular star polyhedra, projective geometry, and topology. Lazare Carnot called projective geometry "Geometrie de position" (1803). Ch. Staudt in "Geometrie der Lage" (1847) showed that the essence of projective geometry is the study of mutual arrangement of points, lines, and planes. Louis Poinsot contributed a lot to the development of analysis situs. In the Institut de France in 1809, he read a report "Sur les polygones et les polyedres" [1]. Poinsot conducts his research within the framework of the geometry of the situation, the founder of which was Leibniz. Poinsot builds a regular star polyhedron. Since the time of the Greeks, five types of convex polyhedra were known: a cube, a tetrahedron, an octahedron, a dodecahedron, an icosahedron. Kepler J. in his main work "Harmonice mundi libri V" (1619) pointed out that two more types of regular polyhedra exist, both of which are star polyhedra. Poinsot rediscovered Kepler star polyhedra and constructed two more regular star polyhedra: a large dodecahedron and a large icosahedron. After the publication of the work by Poinsot, the regular stellating polyhedra became known as Kepler-Poinsot regular bodies. A. Cauchy (1813) in the "Recherches sur les Polyedres" proved that the regular Kepler-Poinsot bodies complete the list of all regular stellating polyhedra. In 1882 Stringham found all six regular convex polyhedra of four-dimensional space.

In 35 years, Poinsot turned to the ideas of the "Memoire" of 1810 again. In his work "Reflexions sur les priencipes fondamentaux de la theorie des nombres" [2], he gives an overview of some new ideas in mathematics and notes that the achievements of modern geometry and number theory go hand in hand. If number theory considers numbers in themselves, studies their properties, independent of the way they are represented and acted on, and rational algebra (or universal arithmetic), starting from ordinary numbers (rational numbers), extends to anything, the higher algebra, in turn, in the theory of equations is predicated on the theory of order and combinations. Geometry – the science of spatial or extended forms – like algebra is divided into two parts: one part studies proportionality and measurement, the other part considers the order and arrangement of objects in space, regardless of their size and shape. This branch of geometry is called the geometry of position, which, according to Poinsot, includes the theory of stellating polygons and polyhedra.

5. Leibniz and combinatorics

Leibniz's idea of analysis situs laid foundation for the theory of configurations, both geometric and combinatorial. Combinations, arrangements, and permutations are the simplest combinatorial configurations. The combination is the basic operation in the theory of configurations. It is not strictly mathematical, but belongs to general intellectual abilities, such as the abilities to classify and create. The essence of combinatorics was first discovered by Leibniz, who also described its area of application. Leibniz regarded combinatorics as part of the art of invention. He performed his first combinatorial calculations in 1666 in his dissertation "Dissertatio de arte combinatoria", and then throughout his life repeatedly returned to reflections on the role of combinatorics in the system of scientific knowledge.

Leibniz's views on the high importance of combinatorial art were shared by J.J. Sylvester. Sylvester wrote several articles on combinatorial configurations, the first one being the article of 1844 "Elementary researches in the analysis of combinatorial aggregation", in which he discussed the rules for the formation of different sets and systems of sets from elements of a given n-set. He wrote: "The present theory may be considered as belonging to a part of mathematics which bears to the combinatorial analysis much the same relation as the geometry of position to that of measure, or the theory of numbers to computative arithmetic; number, place, and combination being the three intersecting but distinct spheres of thought which all mathematical ideas admit of being referred" [3]. To the ideas of 1844, Sylvester returned in 1861: "I have elsewhere given the general name of Tactic to the third pure mathematical science, of which order is the proper sphere, as is number and space of the other two. Syntax and Groups are each of them only special branches of tactic" [4], and in "Concluding paper on Tactic": "Tactic appears to me to constitute the main stem from which all others, including even arithmetic itself, are derived and secondary branches. The key to success in dealing with the problems of this incipient science (as I suppose of most others) must be sought for in the construction of an apt and expressive notation, and in the discovery of language by force of which the mind may be enabled to lay hold of complex operations and mould them into simple and easily transmissible forms of thought" [5]. However, Sylvester did not realize this wide idea, limited to solving particular problems. A. Cayley shared Sylvester's views on tactic. In 1864, Cayley proposed in his article "On the notion and boundaries of algebra" to distinguish between two types of operations in algebra: tactical and logistic: "Although it may not be possible absolutely to separate the tactical and the logistical operations; for in (at all events) a series of logical operations, there is always something that is tactical, and in many tactical operations (e.g.in the Partition of Numbers) there is something which is logistical, yet the two great divisions of Algebra are Tactic and Logistic. Or if, as might be done, we separate Tactic off altogether from Algebra, making it a distinct branch of Mathematical Science, then (assuming in Algebra a knowledge of all the Tactic which is required) Algebra will be nothing else than Logistic" [6]. In 1896 the American mathematician Moore E.H. in the article "Tactical memoranda" [7] introduced the term "tactical configuration". In his article Moore considers numerous examples of tactical systems and proves their properties. A generalization of the concept of "tactical configuration" in XX was the concept of a block design.

6. Topology and graph theory

In 1895 Henry Poincare published his topological work "Analysis Situs". A new subdiscipline in mathematics was born. In the introduction to his first major topology paper, the Analysis situs, Poincare (1895) announced his goal of creating an n-dimensional geometry:"...geometry is the art of reasoning well from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset". Because "positions must not be upset", Poincare sought what Leibniz called Analysis situs, a geometry of position, or what we now call topology. He cited as precedents the work of Riemann and Betti, and his own experience with differential equations, celestial mechanics, and discontinuous groups.

The founder of the graph theory is Euler, who in 1736 published a solution to the problem of the Konigsberg bridges. Only 200 years later appeared Denes Konig's textbook "Theorie der endlichen und unendlichen Graphen" (1936). Konig's book was a major factor in the growth of interest in graph theory worldwide. It was eventually translated into English. Many of the topics dealt with by Konig are to be found in almost any text on the subject, for example, Euler trails, Hamiltonian cycles, mazes, trees, directed graphs and factorisations, but unlike many later authors, Konig considers infinite as well as finite graphs. In the Preface to his book, Konig discusses whether graph theory is a branch of topology or a branch of combinatorics. He argues that it is the latter:"... mainly because we attribute to the elements of a graph - vertices and edges - no geometrical content at all: the vertices are arbitrary distinguishable elements, and an edge is nothing other than a unification of its two endpoints. This abstract point of view - which Sylvester emphasized in 1873 - will be strictly maintained in our representation, with the exception of some examples and applications". Serious development of graph theory was in the second half of the 20th century.

7. Conclusions

Leibniz's idea of Analysis Situs contributed to the development of geometry, combinatorics, the emergence of graph theory and the creation of topology in the 19th century. Combinatorics and graph theory in the twentieth century formed the core of discrete mathematics. In connection with powerful development of computer technology, discrete mathematics is now becoming the most demanded section of mathematics and demonstrates the true triumph of Leibniz's ideas

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