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# Generalized Stein equation on extended class of functions

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**Abstract.** The Stein method is a powerful tool of obtaining convergence rates in limit theorems of probability theory and thus is widely used in various contexts. The deep results on this way were established by C.Stein, L.Chen, L.Goldstein, G.Reinert, Q-M.Shao, I.Tyurin and other researchers. To employ the mentioned method one has to derive (and study) the Stein equation for distribution $P$ of a random variable $X$. It involves, along with a specified operator $T$ defined on a class of functions $\mathcal{F}$, the law of another random variable $Y$ as well (to evaluate distance, in a sense, between the laws of $X$ and $Y$). The aim of our work is to provide a generalization of the Stein equation allowing to use the functions $f$ for which the classical Stein's identity is not satisfied. We also show that our equation permits to characterize the law of $X$, i.e. this equation holds for all $f$ in the extended $\mathcal{F}$ if and only if $Law(Y) = P$. In the operator theory framework one can observe that a so-called density approach is a particular case of our method. Due to the modification proposed we can estimate a distance between the target distribution $P$ of a random variable $X$ and a distribution of $Y$ when $supp(Y)$ is not a subset of $supp(X)$. Moreover, we can write the characteristic Stein equations for random variables with non-interval support. We illustrate some advantages of the introduced method by a number of interesting examples.

**Keywords:** the Stein equation, Stein's identity, density approach, Renyi theorem.

## 1. Introduction

The Stein method is a technique to estimate the distance between cumulative distribution functions proposed by C. Stein in 1972 (see [1]) in the context of normal approximation. He considered the normalized sums of $m$-dependent random variables to establish the convergence rate in the corresponding central limit theorem. Later this method was adapted to approximation with a number of other probability distributions, such as Poisson, exponential, binomial distributions and etc. The important results were obtained by L. Chen, A. Barbour, A.N. Tikhomirov, L. Goldstein, G. Reinart, I.S. Tyurin and other researchers. More specific information concerning the main results can be found in the paper [**?**].

Let us recall some key points of the method mentioned above.

Firstly, one must select a so-called *target* distribution, a distance to which will be estimated. Suppose random variable $X$ has the target distribution. The chosen law of $X$ is associated with an operator $\mathcal{T}_X$ such that *the Stein equation*

$$\mathcal{T}_X f(x) = h(x) - \mathbb{E}h(X) \tag{1}$$

has a solution $f$ for each function $h$ from a set $\mathcal{H}$. Then for a random variable $Y$, function $h \in \mathcal{H}$ and corresponding solution $f$ of the equation (1), it follows that

$$\mathbb{E}\mathcal{T}_X f(Y) = \mathbb{E}h(Y) - \mathbb{E}h(X). \tag{2}$$

We assume the mathematical expectations above exist. The class of functions $\mathcal{H}$ should be chosen in such a way that the supremum over $\mathcal{H}$ for the modulus of the right-hand side of (2) determines a metric to find the distance between distributions of $X$ and $Y$. This construction proved to be very successful, since it allows the researchers to obtain results for the Kolmogorov, Kantorovich and total variation distances, some ideal metrics (of order not more than one) and others.

In order to find a candidate for operator $T_X$, various approaches (such as the method of antisymmetric functions and the $L^2$-approach) were developed to write down the so-called *Stein identity*

$$\mathbb{E}\mathcal{T}_X f(X) = 0$$

for a wide class $\mathcal{F}$ of functions $f$. One can apply Stein's method using such an operator if a solution of the Stein equation exists and lies in $\mathcal{F}$ for each function $h$ from the set $\mathcal{H}$. Instead, in a number of papers, authors have checked the necessary condition to apply the Stein method:

$$\mathbb{E}\mathcal{T}_X f(Y) = 0 \quad \forall f \in \mathcal{F} \;\Leftrightarrow\; Law(X) = Law(Y). \tag{3}$$

## 2. Generalized Stein equation

In the paper of Ley et al. [3] the operators

$$\mathcal{T}_X f(x) := \frac{D(f(x)p_X(x))}{p_X(x)}$$

are considered. Here $p_X$ is the density of $X$ by some measure $\mu$ (usually the Lebesgue measure or the counting measure on a lattice), the linear operator $D$ has a right inverse operator and does not depend on the density $p_X$. Under a number of additional assumptions, the statement (3) ("characterization") is proved for this type of operators in the above-mentioned article [3].

It is important to notice that the condition $supp(Y) \subseteq supp(X)$ must be satisfied to estimate the distance between the random variables $X$ and $Y$ using operators from Ley et al. [3]. It follows from the next fact for the proposed operators: the equation (1) is satisfied only on the support of the random variable $X$ that has the target distribution.

In order to generalize the method, consider an equation

$$\mathbb{E}\mathcal{T}_p f(Y) - \mathbb{E}\mathcal{T}_p f(X) = \mathbb{E}h(Y) - \mathbb{E}h(X) \tag{4}$$

for an extended class of functions $f$ on which the Stein identity may not be satisfied, but for these functions the corresponding averaged equation holds.

One can suppose $Df = f'$ in the equation (4) for any random variable with continuously differentiable density and find a family of Stein equations, which coincides with the equations derived in [4] by the density approach.

Employing new equations of the type (4) we can
1) remove the condition $supp(Y) \subseteq supp(X)$,
2) consider $X$ with non-interval support by Lebesgue measure or counting measure on a lattice.

An example to the first point is as follows: an analogue of Renyi's theorem for random variables with non-positive support is proved.

**Theorem 1.** *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of $m$-dependent random variables. Suppose $\mathbb{E}Y_i = \frac{1}{\lambda}$ and $supp(Y_i) \subset [a, b]$, $\forall i \in \mathbb{N}$. Assume that $N_p$ is independent of $\{Y_i\}_{i=1}^{\infty}$ and has geometric distribution with parameter $p$. Then $W_p = \frac{p}{1-p}\sum_{i=1}^{N_p} Y_i$ satisfies*

$$W_p \xrightarrow{d} Z \sim Exp(\lambda), \quad p \to 0.$$

More details on Renyi's theorem could be found in [5]. In addition, one can find the similar result for non-negative variables in the article [6].

Now let us to illustrate the second point. Consider lattice distributions with a step $\delta$. The equation

$$\mathbb{E}\Big[f(Y)\frac{p_X(Y+\delta)}{p_X(Y)} - f(Y-\delta)\Big] = 0$$

characterizes random variables with these distributions in the sense of (3) if there isn't a point $x_0$ on lattice such that $p_X(x_0) = 0$ and there exist non-zero values $p_X(x)$ for arguments both less and greater than $x_0$.

For continuous case, one can consider the equation (4) for $X$ with the density $p(x) = \frac{\lambda}{2}e^{\lambda(a-|x|)}$ on the support $(-\infty, -a] \cup [a, \infty)$. Thus, (4) is turned into

$$\mathbb{E}\big[f'(Y) - \lambda\, sign(Y) f(Y)\big] - \frac{\lambda}{2}e^{-\lambda a}(f(-a) - f(a)) = \mathbb{E}h(Y) - \mathbb{E}h(X). \tag{5}$$

It is possible to show the left-hand side of (5) characterizes random variables with above-mentioned density in the sense of the condition (3).

In this report we discuss the development of a new approach to aviod problems associated with type of support.

## 3. Characterization involving factorial moment generating functions

Let us turn to the Stein-Tikhomirov method based on determining a characteristic function by a differential equation. For instance, in the paper [7] nonclassical central limit theorem with new conditions was proved using the equation

$$f'(t) = -\sigma^2 t f(t).$$

In our work a similar method is developed for factorial moment generating functions. For a Poisson distribution with parameter $\lambda$, generating function $f(z) = e^{\lambda(z-1)}, z \in [0,1]$, is the unique solution of the Cauchy problem

$$\begin{cases} f' - \lambda f = 0, \\ f(1) = 1. \end{cases}$$

We introduce the operator

$$A_\lambda f = f' - \lambda f,$$

where $\lambda = \frac{d}{dt} f(t)\Big|_{t=1} = \mathbb{E}\xi$. Thus, $\lambda$ is equal to the mathematical expectation of a random variable, which produces argument for operator $A_\lambda$. So, $\lambda$ under the operator can be omitted.

Equation

$$Af(t) \equiv 0 \text{ for } t \in [0,1]$$

characterizes the factorial moment generating function of a Poisson random variable in a class of random variables with the same mathematical expectation.

Application of the operator leads to the following new result.

**Theorem 2.** *Let random variables $\{\xi_{n,k}\}_{k=\overline{1,n}}^{n}$ be a triangular array of row-independent random variables with supports in $\mathbb{Z}_+$ and satisfy the condition $\sum_{k=1}^{n} \mathbb{E}\xi_{n,k} \underset{n\to\infty}{\to} \lambda < \infty$.*

*Then*

*1)* $$\lim_{n\to\infty} \sum_{k=1}^{n} |Af_{n,k}(t)| = 0 \quad \Rightarrow \quad \sum_{k=1}^{n} \xi_{n,k} \underset{n\to\infty}{\overset{d}{\to}} \xi \sim Pois(\lambda),$$

*2)* $$\begin{matrix} \sum_{k=1}^{n} \xi_{n,k} \underset{n\to\infty}{\overset{d}{\to}} \xi \sim Pois(\lambda) \\ \max_{k=\overline{1,n}} \mathbb{E}\xi_{n,k} \underset{n\to\infty}{\to} 0 \end{matrix} \quad \Rightarrow \quad \lim_{n\to\infty} \sum_{k=1}^{n} |Af_{n,k}(t)| = 0,$$

*where $f_{n,k}$ - factorial moment generating function of $\xi_{n,k}$.*

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