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Diffusion Approximation of Branching Processes

N. Limnios*

* *Sorbonne University, Université de technologie de Compiègne*
LMAC Laboratory of Applied Mathematics of Compiègne -
CS 60319 - 60203 Compiègne cedex, France (E-mail: nlimnios@utc.fr)

Abstract. We consider Bienaymé-Galton-Watson (BGW) and continuous-time Markov branching processes and prove diffusion approximation results in the near critical case. In particular, we give new proofs and derive necessary and sufficient conditions for diffusion approximation to hold of Feller-Jiřina and Jagers theorems. The proofs are new and are not based on generating function theory.

Keywords: Branching process, diffusion approximation, near critical case.

1. Introduction

In the present work we study diffusion approximation of near critical Markov branching processes in discrete-time (BGW), and continuous-time Markov age-dependent branching processes, (see, e.g., [2, 5, 6, 9]). Especially, Feller-Jiřina theorem ([1, 4, 10]) and Jagers theorem ([9]) are revisited. We present a different method to obtain diffusion approximation based on Markov generators convergence [3, 11] and semimartingale relative compactness [12], (see also [8, 13]). Moreover, we prove that the near critical condition is a necessary and sufficient condition for diffusion approximation of a Markov branching process to hold.

The main results are presented in Section 2; the main step of proofs in Section 3, and finally a short conclusion is also given in the last section.

2. Diffusion approximation results

Consider a Bienamé-Galton-Watson branching process in discrete-time

$$Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{n-1,j}, \quad n \geq 1, \quad Z_0 = 1.$$

Denote by $\mu := \mathbf{E}\xi_{n-1,j}$ and $\sigma^2 := \mathrm{Var}\,(\xi_{n-1,j})$, the common mean and variance of offspring, $\xi_{n-1,j}$, which are i.i.d. random variables. We denote also ξ_j instead of $\xi_{n,j}$ in some places.

Define now the family of processes in series scheme, indexed by the series parameter $\varepsilon > 0$, say Z_n^ε, and define the processes $Y_t^\varepsilon := \varepsilon Z_{[t/\varepsilon]}^\varepsilon$, $t \geq 0$, $\varepsilon > 0$. We denote by ξ_j^ε, $\varepsilon > 0$ the number of offspring.

The following assumptions are needed in the sequel.

C1: Offspring mean assumption: $\mu_\varepsilon = 1 + \varepsilon\alpha + o(\varepsilon)$, as $\varepsilon \downarrow 0$, and $\alpha \in \mathbb{R}$ a constant;

C2: Offspring variance assumption: $\sigma_\varepsilon^2 = \sigma^2 + o_\varepsilon(1)$, with $0 < \sigma^2 < \infty$.

C3: Initial value assumption: Y_0^ε converge to a point $x \in \mathbf{R}$, as $\varepsilon \downarrow 0$.

What we mean by the *near critical* hypothesis can be expressed by assumptions C1. The near critical case is a necessary and sufficient condition for a BGW process to give a diffusion approximation. Let $C_0^2(\mathbf{R})$ be the space of real-valued functions defined on $\mathbb{R}$ twice continuously differentiable vanishing at infinity. Let $\Longrightarrow$ mean the weak convergence in the Skorohod space $D_\mathbb{R}[0,\infty)$.

Theorem 1 *Conditions C1-C3 are necessary and sufficient that the following weak convergence holds*

$$Y_t^\varepsilon \Longrightarrow W_t, \qquad as \quad \varepsilon \downarrow 0,$$

where W_t is a diffusion process defined by the generator $L\varphi(x) = \alpha x\varphi'(x) + \frac{1}{2}\sigma^2 x\varphi''(x)$, with initial value $W_0 = x$, and $\varphi \in C_0^2(\mathbf{R})$. Here φ' and φ'' are the first and second derivative of the function φ.

The "if" part of the above theorem is the well known Feller-Jiřina theorem ([10]). We will give here another proof, without generating function.

Let us now consider Markov branching processes in continuous time, $Z_t, t \in \mathbb{R}_+$, for $t, s \geq 0$,

$$Z_{t+s} = \sum_{i=1}^{Z_t} \xi_i^{(s)}$$

where $\xi_i^{(s)}$ is the number of offspring of the i-th particle living in time t. The particle lifetime follows an exponential distribution with mean $1/\lambda$, $\lambda > 0$. Let us consider the family of processes $Y_t^\varepsilon := \varepsilon Z_{t/\varepsilon}^\varepsilon$, $t \geq 0$, $\varepsilon > 0$. Then we have the following result.

Theorem 2 *Conditions C1-C3 are necessary and sufficient that the following weak convergence holds*

$$Y_t^\varepsilon \Longrightarrow W_t, \qquad as \quad \varepsilon \downarrow 0,$$

where W_t is a diffusion process defined by the generator $L\varphi(x) = \alpha\lambda x\varphi'(x) + \frac{1}{2}\lambda\sigma^2 x\varphi''(x)$, with initial value $W_0 = x$, and $\varphi \in C_0^2(\mathbf{R})$.

The "if" part of the above theorem is the well known Jagers theorem ([9]). We will give here another proof, without generating function.

Remark. If we replace condition C2 by $\sigma_\varepsilon^2 = o_\varepsilon(1)$, then we get, for both Theorems, as limit process, W_t, the deterministic function e^{bt}, with $b = \alpha$, for Theorem 1, and $b = \lambda\alpha$, for Theorem 2. This provides an average approximation for the considered branching processes.

3. Proofs

Let us give the main steps of proofs of the above two theorems.
Proof of Theorem 1. For any $\varepsilon > 0$, the process Y_t^ε is a Markov process with state space $E_\varepsilon = \varepsilon\mathbb{N}$, $\mathbb{N} := \{0, 1, 2...\}$ and semigroup operator P_ε, defined by $P_\varepsilon\varphi(x) = \mathbb{E}\varphi(\varepsilon\sum_{j=1}^{x/\varepsilon}\xi_j^\varepsilon)$. Set $S_n := \sum_{j=1}^n(\xi_j^\varepsilon - 1)$.

The discrete generator $\mathbb{L}^\varepsilon := \varepsilon^{-1}(P_\varepsilon - I)$ can be written in asymptotic form, for $x \in E_\varepsilon$ and $\varphi \in C_0^2(\mathbb{R})$, as

$$\mathbb{L}^\varepsilon\varphi(x) = \varepsilon^{-1}x(\mu_\varepsilon - 1)\varphi'(x) + \frac{1}{2}[\sigma_\varepsilon^2 x + \varepsilon^{-1}x^2(\mu_\varepsilon - 1)^2]\varphi''(x) + \theta^\varepsilon(x).$$

where

$$|\theta^\varepsilon(x)| \leq \frac{\varepsilon}{2}(x\sigma^2 + x^2a^2 + o_\varepsilon(1))\mathbb{E}w(\varphi'', \varepsilon S_{x/\varepsilon}).$$

The function $w(\varphi'', \delta)$ is the modulus of continuity of the continuous function φ'', and we have by bounded convergence theorem that

$$\mathbb{E}w(\varphi'', \varepsilon S_{x/\varepsilon}) \to 0, \quad \varepsilon \to 0.$$

Introducing the near critical conditions to the above generator we get the announced limit generator. Now for the relative compactness, we will prove the following two facts ([12]). First the compact containment condition has to be proved

$$\lim_{c\to\infty}\sup_{0<\varepsilon<\varepsilon_0}\mathbb{P}\Big(\sup_{0\leq t\leq T}|Y_t^\varepsilon| > c\Big) = 0, \tag{2}$$

and second the inequality $\mathbf{E}|Y_t^\varepsilon - Y_s^\varepsilon|^2 \leq k|t-s|$. The following result holds.

Lemma 1 *The following inequality holds*

$$\mathbf{E}\sup_{0\leq t\leq T}|Y_t^\varepsilon|^2 \leq K_T,$$

where K_T is a constant depending on T and independent of ε.

Now from Lemma 1 and the Kolmogorov inequality we get the compact containment condition (2). On the other hand we take, for $s < t$,

$$\mathbf{E}\Big[\varepsilon^2(Z_{[t/\varepsilon]}^\varepsilon - Z_{[s/\varepsilon]}^\varepsilon)^2\Big] \leq \varepsilon([t/\varepsilon] - [s/\varepsilon])\sigma^2 \sim |t-s|\sigma^2.$$

The conclusion now is clear.

Proof of Theorem 2. The generator $\mathbb{L}^\varepsilon\varphi(x) := \varepsilon^{-1}\lim_{s\to 0}\{\mathbf{E}\varphi(Y_s^\varepsilon) - \varphi(x)\}/s$ can be written in asymptotic form as follows, for $x \in E_\varepsilon$ and $\varphi \in C_0^2(\mathbb{R})$,

$$\mathbb{L}^\varepsilon\varphi(x) = \varepsilon^{-1}\lambda x(\mu_\varepsilon - 1)\varphi'(x) + \frac{1}{2}\lambda x[\sigma_\varepsilon^2 + (\mu_\varepsilon - 1)^2]\varphi''(x) + \theta^\varepsilon(x).$$

and introducing the near critical conditions, we get

$$\mathbb{L}^\varepsilon\varphi(x) = \lambda(\alpha + o_\varepsilon(1))x\varphi'(x) + \frac{1}{2}(\sigma^2 + o_\varepsilon(1))\lambda x\varphi''(x) + \theta^\varepsilon(x).$$

The relative compactness is proved in the same way as in Theorem 1, and the conclusion follows.

4. Conclusions

In this work we presented diffusion approximation results for Markov branching processes by a new way without generating function support. Moreover, near critical condition are proved to be a necessary and sufficient condition for diffusion approximation for the considered branching processes.

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