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Limit theorems for infinite-channel queueing systems with heavy-tailed service times

E.E. Bashtova*, E.A. Chernavskaya*

* *Department of Probability Theory,*
Moscow State University,
Leninskie Gory 1, Moscow, 119991, Russia

Abstract. We consider an infinite-channel queueing system with a regenerative input flow. The service times are heavy-tailed, so that mathematical expectation of service time is infinite. Therefore the number of customers in the system increases with time.We investigate asymptotic behavior of the number of servers busy at time t. We present a functional limit theorem that characterize growth of this characteristics of the system.

Keywords: analytical methods in probability theory, queueing theory, limit theorems, point processes, random sums.

1. Introduction

We consider an infinite-channel queueing system with a regenerative input flow $X(t)$. We investigate asymptotic behavior of the process $q(t)$ that is the number of servers busy at time t, $t \geq 0$.

This model appears in many contexts other than queueing theory. One example (for recurrent flow) is the number of colonies still in existence at time t in a branching process with immigration. More generally, one can consider a point process, where each point undergoes an independent translation forward in time.

The service times are heavy-tailed, so that mathematical expectation of service time is infinite. Because of this, the number of customers in the system increases with time. Such model with recurrent flow was considered in [2]. And one-dimensional analogue of central limit theorem for number of customers in a system $GI|G|\infty$ was proved there. We consider regenerative input flow and prove analogue of functional central limit theorem. Regenerative flow includes many types of flows considering in queueing theory. For example, recurrent, Markov-modulated, Markov Arrival flows are regenerative [1]

2. Main result

To begin with we give the definition of the regenerative flow.

Definition 1. The stochastic flow $X(t)$ is called regenerative if

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— There exists filtration $\{\mathcal{F}^X_{\le t}\}_{t\ge 0}$ such that $X(t)$ is measurable with respect $\{\mathcal{F}^X_{\le t}\}_{t\ge 0}$.
— There is an increasing sequence of Markov moments $\{\theta_j, j\ge 0\}(\theta_0 = 0)$ with respect to $\{\mathcal{F}^X_{\le t}, t\ge 0\}$ such that the sequence

$$\{\varkappa_j\}_{j=1}^{\infty} = \{X(\theta_{j-1}+t) - X(\theta_{j-1}), \theta_j - \theta_{j-1}, t \in (0, \theta_j - \theta_{j-1}]\}_{j=1}^{\infty}$$

consists of independent identically distributed (iid) random elements.

Then $\{\theta_j\}_{j=1}^{\infty}$ are called regeneration points of $X(t)$ and $\tau_j = \theta_j - \theta_{j-1}$, $(\theta_0 = 0)$ regeneration periods.

Service times $\{\eta_j\}_{j=1}^{\infty}$ form the sequence of independent identically distributed random variables with distribution function $B(x)$. We assume that the following condition is fulfilled

$$1 - B(t) = \overline{B}(t) \sim \frac{\mathcal{L}(t)}{t^{\Delta}},\ 0 < \Delta < 1$$

as $t\to\infty$, where $\mathcal{L}(t)$ is slowly varying function as $t\to\infty$.

It means that $\overline{B}(t)$ is regularly varying function. Note that condition (2) implies that

$$\beta(t) = \int_0^t \overline{B}(y)dy \to \infty,\ t\to\infty$$

and so, the mathematical expectation of of service times is *infinite.*

We investigate the growth of $q(t)$ that equals number of customers in the system at time t. Then

$$q(t) = \sum_{i=1}^{X(t)} \mathbb{I}\left(\eta_i > t - t_i\right)$$

Denote

$$\xi_i = X\left(\theta_i\right) - X\left(\theta_{i-1}\right), \quad \tau_i = \theta_i - \theta_{i-1},\ i\ge 1,$$

$$\lambda = \frac{\mathsf{E}\xi_1}{\mathsf{E}\tau_1},$$

$$\beta(t) = \int\limits_0^t \overline{B}(x)dx \sim \mathcal{L}(t)t^{1-\Delta}$$

Theorem 1. Let $\mathsf{E}\tau_1^3 < \infty$, $\mathsf{E}\xi_1^4 < \infty$. Then the sequence of processes

$$q_T^*(t) = \frac{q(tT) - \lambda\beta(tT)}{\sqrt{\mathcal{L}(T)T^{1-\Delta}}}, \quad t \in [0, h]$$

C-converges as $T \to \infty$ to a gaussian process with zero mean and correlation function

$$R(t, t+u) = \frac{\lambda}{1-\Delta}((t+u)^{1-\Delta} - u^{1-\Delta}), \quad u \geq 0, t, t+u \in [0, h]$$

Theorem 1 is a functional limit theorem for sums of random variables with a random summation index. The proof of the theorem is greatly complicated by the fact that the process $q(t)$is the sum of a random number of dependent summands. And the summation index also depends on the terms.

We first prove the theorem on the convergence of finite-dimensional distributions, and then we verify the condition of the density of measures.

Proofs of both parts consist of several stages, such as majorization, various estimates, and the application of the demi-martingale theory [3].

References

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