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On mean-field $GI/GI/1$ queueing model: existence and uniqueness

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Abstract. A mean-field extension of the queueing system $GI/GI/1$ is considered. Under certain assumptions on intensities, the process is constructed as a Markov solution of a martingale problem.

Keywords: GI/GI/1, mean-field process, intensities, existence, uniqueness.

1. Introduction

Mean-field approach in the theory of queueing systems is designed to take into consideration large interacting ensembles of queues by replacing these interactions by a suitable "mean field". In particular, this approach showed rather fruitful in systems with countable state spaces. In this work we propose a method of constructing a more general extension of the system $GI/GI/1$ (more precisely, $GI/GI/1/\infty$) under certain restrictions on intensities of arrivals and service, which intensities may depend on the state as well as on the marginal distribution of the process. Weak uniqueness will be also discussed.

2. Main section

The state space of the process under consideration is the union

$$\mathcal{X} := (0, x) \cup \bigcup_{n=1}^{\infty} (n, x, y), \quad x, y \geq 0.$$

The meaning of n here is the number of "customers" in the system; the value x stands for the elapsed time from the last arrival, while y signifies the elapsed time of the current service. There is only one server which works without breaks (if there is at least one customer in the system) and it is always in a working state. All newly arrived customers stand in a queue of the infinite capacity, and for simplicity only we assume the FIFO discipline of service. It is assumed that at any time t at any state $X = (n, x, y)$ (or $X = (0, x)$ for $n = 0$) there are *intensities* of service $h[t, X_t, \mu_t]$ and arrivals $\lambda[t, X_t, \mu_t]$, where μ_t is the distribution of the random variabe X_t itself.

Note that occasionally we will be using notation $(0, x, y)$ where y is a "false" variable. The process is linear–deterministic Markov one (see [1, 3]).

The assumptions:

(A1)

$$\lambda[t, X, \mu] = \int \lambda(t, X, Y)\mu(dY), \quad h[t, X, \mu] = \int h(t, X, Y)\mu(dY)$$

(A2) The functions $\lambda(t, X, Y)$ and $h(t, X, Y)$ are Borel and bounded.
(A3) The functions $\lambda(t, X, Y)$ and $h(t, X, Y)$ are continuous in all variables.
(A4) The functions $\lambda(t, X, Y)$ and $h(t, X, Y)$ are bounded away from zero.

For $X \in \mathcal{X}$ let us denote (X^- is not defined for $X = (0, x)$)

$$X^+ := (n + 1, 0, y), \quad \text{for } X = (n, x, y),$$
$$X^- := 1(n > 0)(n - 1, x, 0), \quad \text{for } X = (n, x, y),\ n \geq 1.$$

Theorem 1 *Let the assumptions (A1)–(A3) be satisfied. Then for any fixed $X_0 \in \mathcal{X}$, on some probability space there exists a Markov process $(X_t,\, t \geq 0)$ with marginal distributions μ_t and intensities $\lambda[t, X_t, \mu_t]$, $h[t, X_t, \mu_t]$; in other words, such that for any bounded continuous function $g(X)$ with bounded continuous derivatives in (x, y), the expression*

$$M_t := g(X_t) - g(X_0) - \int_0^t L(s, X_s, \mu_s)g(X_s)\, ds \tag{1}$$

is a martingale, where for $X = (n, x, y)$, $X' = (n', x', y')$, $n \geq 0$, $t \geq 0$,

$$\begin{aligned} L(t, X', \mu)g(X) &:= \lambda[t, X', \mu](g(X^+) - g(X)) \\ &+ 1(n > 0)h[t, X', \mu](g(X^-) - g(X)) \\ &+ \frac{\partial}{\partial x}g(n, x, y) + 1(n > 0)\frac{\partial}{\partial y}g(n, x, y). \end{aligned} \tag{2}$$

For any given measure-valued function $(\mu_s,\, s \geq 0)$ in $L(s, X_s, \mu_s)$, the martingale problem *(see [4]) (1) has a weakly unique solution.*

Equivalently to (1), *Dynkin's indentity* holds true for any function $g(X)$ from the same class,

$$\mathbb{E}_{0,X_0} g(X_t) = g(X_0) + \mathbb{E}_{0,X_0} \int_0^t L(s, X_s, \mu_s)g(X_s)\, ds.$$

Moreover, equivalently, for any $0 \le t_1 < t_2 \ldots < t_{m+1}$, and for any Borel bounded functions $\phi_k(X)$, $X \in \mathcal{X}$,

$$\mathbb{E}_{0,X_0}\left(g(X_{t_{m+1}}) - g(X_{t_m}) - \int\limits_{t_m}^{t_{m+1}} L(s, X_s, \mu_s) g(X_s)\, ds\right) \prod_{k=1}^{m} \phi_k(X_{t_k}) = 0. \tag{3}$$

The equation (3) may be also called a *martingale problem.*

Proof of Theorem 1. For any $n \ge 1$ consider a process (X^n_t), with initial data $X^{n,\delta}_0 = X_0$ and intensities of jumps up and down, respectively,

$$\lambda[t, X^n_{t-1/n}, \mu^n_{t-1/n}], \quad h[t, X^n_{t-1/n}, \mu^n_{t-1/n}].$$

where X^n_t with $t < 0$ is understood as X^n_0, and similarly for μ^n_t. The process (X^n_t) for each n are constructed by induction successfully on the intervals $[0, 1/n], [1/n, 2/n]$, etc. Due to the boundedness assumption on both intensities, there is no blow up and the processes for any n are defined for any $t \ge 0$ as càdlàg pure jump processes. Moreover, for any t probability of jump at t for any X^n equals zero. The processes $(X^n_t,\, t \ge 0)$ for $n \ge 1$ being constructed, let us introduce on some probability space *independent* equivalent processes $(\xi^n_t,\, t \ge 0)$; let $\mathbb{E}'$ stand in all cases for the integration with respect to the *third variable*, e.g.,

$$\mathbb{E}' h(t, X^n_t, \xi^n_t) := \int h(t, X^n_t, y) \mu^n_t(dy).$$

It can be checked that the assumptions of the Lemma 1 from the Appendix are satisfied. Hence, on some new probability space there are equivalent – and, hence, *Markov with the same generators* – processes $(\tilde{X}^n_t, \tilde{\xi}^n_t)$ and a limiting pair $(\tilde{X}_t, \tilde{\xi}_t)$ such that for some subsequence $(\tilde{X}^{n'}_t, \tilde{\xi}^{n'}_t) \stackrel{\mathbb{P}}{\to} (\tilde{X}_t, \tilde{\xi}_t)$, $n' \to \infty$, for each t. It follows due to the boundedness of all intensities that the limiting process $(\tilde{X}_t, \tilde{\xi}_t)$ is also stochastically continuous. More than that, it is a pure jump process with a finite number of jumps on any bounded interval with probability one. Moreover, due to $\lim_{h\downarrow 0} \sup_n \sup_{t,s\le T;\, |t-s|\le h} \mathbb{P}(|\tilde{X}^n_t - \tilde{X}^n_s| > \epsilon) = 0$ for any $\epsilon > 0$ it follows that

$$\tilde{X}^{n'}_{t-1/n'} \stackrel{\mathbb{P}}{\to} \tilde{X}_t, \quad n' \to \infty.$$

The analogue of Dynkin's formula (3) reads,

$$\mathbb{E}_{0,X_0}\left[\left(g(\tilde{X}^{n'}_{t_{m+1}}) - g(\tilde{X}^{n'}_{t_m}) - \int\limits_{t_m}^{t_{m+1}} \mathbb{E}' L(s, \tilde{X}^{n'}_{s-1/n'}, \tilde{\xi}^{n'}_{s-1/n'}) g(\tilde{X}^{n'}_s)\, ds\right)\right.$$

$$\times \prod_{k=1}^{m} \phi_k(\tilde{X}^{n'}_{t_k})\Bigg] = 0. \qquad (4)$$

By continuity of λ and h, and due to the stochastic continuity of the processes $\tilde{X}$ and $\tilde{\xi}$, and since all integrand expressions are bounded, and by virtue of Lebesgue's bounded convergence Theorem, we obtain from (4) in the limit with *continuous bounded* functions (ϕ_k),

$$\mathbb{E}_{0,X_0}\left(g(\tilde{X}_{t_{m+1}}) - g(\tilde{X}_{t_m}) - \int\limits_{t_m}^{t_{m+1}} \mathbb{E}' L(s, \tilde{X}_s, \tilde{\xi}_s) g(\tilde{X}_s)\, ds\right) \prod_{k=1}^{m} \phi_k(\tilde{X}_{t_k}) = 0. \qquad (5)$$

Since the distribution of the random variable $\tilde{\xi}_t$ is the same as the one of $\tilde{X}_t$ – let us denote it by $\tilde{\mu}_t$ – then (5) can be equivalently written as

$$\mathbb{E}_{0,X_0}\left(g(\tilde{X}_{t_{m+1}}) - g(\tilde{X}_{t_m}) - \int_{t_m}^{t_{m+1}} Lg(s, \tilde{X}_s, \tilde{\mu}_s)\, ds\right) \prod_{k=1}^{m} \phi_k(\tilde{X}_{t_k}) = 0. \qquad (6)$$

Due to the properties of measures on $\mathbb{R}^d$, the formula (6) holds true for any Borel bounded functions (ϕ_k), too. Due to [3], solution of the "martingale problem" (6) – or, more precisely, of the martingale problem

$$M_t := g(\tilde{X}_{t_{m+1}}) - g(\tilde{X}_{t_m}) - \int_{t_m}^{t_{m+1}} Lg(s, \tilde{X}_s, \tilde{\mu}_s)\, ds \quad \textit{is a martingale},$$

with given $(\tilde{\mu}_s)$ is unique. Hence, by [4, Theorem 4.4.2] the limiting process $\tilde{X}$ is Markov. The required generator (2) with required intensities λ, h follows from (6). This finishes this sketchy proof of the Theorem 1.

3. Uniqueness

Theorem 2 *Let the assumptions (A1)–(A2) and (A4) be satisfied. Then, for any fixed* X_0, *the process* $(X_t,\, t \geq 0)$ *with required intensities* $\lambda[t, x, \mu_t]$ *and* $h[t, x, \mu_t]$ *is unique in distribution.*

Proof of Theorem 2 is based on Girsanov's formula for jump processes (see, e.g., [5]) and will be presented in the full version of this paper.

4. Appendix

Lemma 1 (Skorokhod [6, Ch.1, §6]) *Let ξ^n_t $(t \ge 0,\ n = 0, 1, \ldots)$ be some d-dimensional stochastic processes defined on some probability space and let for any $T > 0$, $\epsilon > 0$ the following hold true:*

$$\lim_{c\to\infty} \sup_n \sup_{t\le T} \mathbb{P}(|\xi^n_t| > c) = 0,$$

$$\lim_{h\downarrow 0} \sup_n \sup_{t,s\le T;\,|t-s|\le h} \mathbb{P}(|\xi^n_t - \xi^n_s| > \epsilon) = 0.$$

Then there exists a subsequence $n' \to \infty$ and a new probability can be constructed with processes $\tilde\xi^{n'}_t$, $t \ge 0$ and $\tilde\xi_t$, $t \ge 0$, such that all finite-dimensional distributions of $\tilde\xi^{n'}_\cdot$ coincide with those of $\xi^{n'}_\cdot$ and such that for any $\epsilon > 0$ and all $t \ge 0$,

$$\mathbb{P}(|\tilde\xi^{n'}_t - \tilde\xi_t| > \epsilon) \to 0, \quad n' \to \infty.$$

5. Conclusions

Under conditions of boundedness and continuity of *intensities* of arrivals and of service in all variables, it is shown that a mean-field Markov process describing the model $GI/GI/1$ does exist. The results will also have important consequences in the mathematical reliability theory, see [2].

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