UDC 519.217.6

Chernoff Approximation of transition kernels of Markov processes

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Abstract. We present a technique to approximate transition kernels of Markov processes, constructed from other, original Markov processes via a random time change. This technique is based on the Chernoff Theorem and leads to the so-called Feynman formulae which can be used for direct calculations and simulation of the considered processes.

Keywords: Approximation of transition kernels, Markov processes, a random time change, Chernoff approximation, Feynman formulae, Feynman-Kac formulae.

1. Introduction

An evolution semigroup $\left(e^{tL}\right)_{t\geq 0}$ with a given generator L (on a given Banach space), on the one hand, allows to solve an initial (or initial-boundary) value problem for the corresponding evolution equation $\frac{\partial f}{\partial t} = Lf$ and, on the other hand, defines the transition probability $P(t,x,dy)$ of an underlying Markov process $(\xi_t)_{t\geq 0}$ (if there is any) through $e^{tL}f(x) = \mathbb{E}^x[f(\xi_t)] = \int f(y)P(t,x,dy)$. We present a technique of approximating evolution semigroups $\left(e^{tL}\right)_{t\geq 0}$ or, respectively, transition kernels $P(t,x,dy)$ by means of the Chernoff Theorem. This theorem provides conditions[1] for a family of bounded linear operators $(F(t))_{t\geq 0}$ to approximate the considered semigroup $\left(e^{tL}\right)_{t\geq 0}$ via the formula $e^{tL} = \lim_{n\to\infty}[F(t/n)]^n$. This formula is called *Chernoff approximation of the semigroup* $\left(e^{tL}\right)_{t\geq 0}$ *by the family* $(F(t))_{t\geq 0}$. And the family $(F(t))_{t\geq 0}$ itself is called *Chernoff equivalent* to this semigroup. If families $(F(t))_{t\geq 0}$ are given explicitly, the expressions $[F(t/n)]^n$ can be directly used for calculations and hence for simulations of underlying stochastic processes. Moreover, if all operators $F(t)$ of a given family $(F(t))_{t\geq 0}$ are integral operators with elementary kernels (or pseudo-differential operators with elementary symbols) the identity $e^{tL} = \lim_{n\to\infty}[F(t/n)]^n$ leads to a representation of the semigroup $\left(e^{tL}\right)_{t\geq 0}$ by limits of n-fold iterated integrals of elementary functions when n tends to infinity. Such representations are called *Feynman formulae*; the limits

[1] These conditions are: $F(0) = \mathrm{Id}$, $\|F(t)\| \leq e^{wt}$ for some $w \geq 0$ and all $t \geq 0$, $\lim_{t\to 0} t^{-1}(F(t)\varphi - \varphi) = L\varphi$ for all $\varphi \in D$, where D is a core for the generator L.

in Feynman formulae usually coincide with functional (path) integrals with respect to probability measures (Feynman–Kac formulae) or with respect to Feynman pseudomeasures (Feynman path integrals). Therefore, the method of Chernoff approximation allows also to establish new Feynman–Kac formulae; different Chernoff approximations (in the form of Feynman formulae) for the same semigroup allow to establish relations between different path integrals. Recently, some Chernoff approximations have been constructed for transition kernels of Markov processes, obtained from some original processes by such procedures as: a random time change, subordination, killing upon leaving a given domain (see [1, 2, 4, 5]). In these constructions, it is supposed that transition kernels of original processes are known or already Chernoff approximated. This assumption holds true, e.g. for Feller processes in $\mathbb{R}^n$ and some Feller diffusions on star graphs and compact Riemannian manifolds (cf. [3, 4, 6]). Below we present a few results, generalizing some constructions in [4] and [5].

2. Chernoff approximation for transition kernels of Markov processes obtained by a random time change

Let Q be a metric space. Let $C_b(Q)$ be the space of bounded continuous functions on Q with supremum-norm $\|f\|_\infty = \sup_{q\in Q}|f(q)|$. Let $C_\infty(Q) := \left\{\varphi \in C_b(Q) : \lim_{\rho(q,q_0)\to\infty}\varphi(q) = 0\right\}$, where q_0 is an arbitrary fixed point of Q and the metric space Q is unbounded with respect to its metric ρ. Let $C_0(Q) := \left\{\varphi \in C_b(Q) : \forall\,\varepsilon > 0\ \exists \text{ a compact } K^\varepsilon_\varphi \subset Q \text{ such that } |\varphi(q)| < \varepsilon \text{ for all } q \notin K^\varepsilon_\varphi\right\}$, where the metric space Q is assumed to be locally compact. Let the Banach space X be any of the spaces $C_b(Q)$, $C_\infty(Q)$, $C_0(Q)$. Let $(T_t)_{t\geq0}$ be a strongly continuous semigroup on X with generator $(L, \mathrm{Dom}(L))$. Consider a function $a \in C_b(Q)$ such that $a(q) > 0$ for all $q \in Q$. Then the space X is invariant under the multiplication operator a, i.e. $a(X) \subset X$. Consider the operator $\widehat{L}$, defined for all $\varphi \in \mathrm{Dom}(\widehat{L})$ and all $q \in Q$ by

$$\widehat{L}\varphi(q) := a(q)(L\varphi)(q), \quad \text{where} \quad \mathrm{Dom}(\widehat{L}) := \mathrm{Dom}(L).$$

We assume that $(\widehat{L}, \mathrm{Dom}(\widehat{L}))$ generates a strongly continuous semigroup (which is denoted by $(\widehat{T}_t)_{t\geq0}$) on the Banach space X. The operator $\widehat{L}$ is called a *multiplicative perturbation* of the generator L and the semigroup $(\widehat{T}_t)_{t\geq0}$, generated by $\widehat{L}$, is called a *semigroup with the multiplicatively perturbed* with the function *a generator*. Some conditions assuring the existence and strong continuity of the semigroup $(\widehat{T}_t)_{t\geq0}$ are discussed in [7, 8].

Theorem 2.1. *Let $(\widehat{L}, \mathrm{Dom}(\widehat{L}))$ be the generator of the strongly continuous semigroup $(\widehat{T}_t)_{t\geq0}$) on X. Let $(F(t))_{t\geq0}$ be a strongly continuous* *family of bounded linear operators on the Banach space X, which is Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$. Consider the family of operators $(\widehat{F}(t))_{t\geq 0}$ defined on X by*

$$\widehat{F}(t)\varphi(q) := (F(a(q)t)\varphi)(q) \quad \text{for all } q \in Q.$$

The operators $\widehat{F}(t)$ act on the space X and the family $(\widehat{F}(t))_{t\geq 0}$ is strongly continuous and Chernoff equivalent to the semigroup $(\widehat{T}_t)_{t\geq 0}$ with multiplicatively perturbed with the function a generator, i.e. the Chernoff approximation

$$\widehat{T}_t\varphi = \lim_{n\to\infty} \left[\widehat{F}(t/n)\right]^n \varphi$$

is valid for all $\varphi \in X$ locally uniformly with respect to $t \geq 0$.

Example 2.2. Let $(X_t)_{t\geq 0}$ be a Markov process with the state space Q and transition probability $P(t,q,dy)$. Let the corresponding semigroup $(T_t)_{t\geq 0}$,

$$T_t\varphi(q) = \mathbb{E}^q\left[\varphi(X_t)\right] \equiv \int\limits_Q \varphi(y)P(t,q,dy),$$

be strongly continuous on the Banach space X, where $X = C_b(Q)$, $X = C_\infty(Q)$ or $X = C_0(Q)$. Then by Theorem 2.1 the family $(\widehat{F}(t))_{t\geq 0}$ defined by

$$\widehat{F}_t\varphi(q) := \int\limits_Q \varphi(y)P(a(q)t,q,dy),$$

is strongly continuous and is Chernoff equuivalent to the semigroup $(\widehat{T}_t)_{t\geq 0}$ with multiplicatively perturbed (with the function a) generator. Therefore, the following Feynman formula is true:

$$\widehat{T}_t\varphi(q_0) = \lim_{n\to\infty} \int\limits_Q \cdots \int\limits_Q \varphi(q_n)P(a(q_0)t/n, q_0, dq_1)P(a(q_1)t/n, q_1, dq_2) \times \cdots$$
$$\times P(a(q_{n-1})t/n, q_{n-1}, dq_n).$$

And the convergence is uniform with respect to $q_0 \in Q$ and locally uniform with respect to $t \in [0,\infty)$.

Remark 2.3. The multiplicative perturbation of the generator L of the Markov process $(X_t)_{t\geq 0}$ with the function a is equivalent to the random time change $X_t \rightsquigarrow X_{\xi_t}$, where $\xi_t := \inf\left\{s > 0 : \int_0^s a^{-1}(X_\tau)d\tau > t\right\}$. Note that $\widehat{P}(t,q,dy) := P(a(q)t,q,dy)$ is not a transition probability any more. Nevertheless, if the transition probability $P(t,q,dy)$ of the original process is known, formula 2.2 allows to approximate the unknown transition probability of the modified process.

Let now $A : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ be a uniformly continuous mapping such that the operator $A(x)$ is symmetric for all $x \in \mathbb{R}^d$, and let there exist $a_0, A_0 \in \mathbb{R}$ such that $0 < a_0 \le A_0 < \infty$ and for all $x, z \in \mathbb{R}^d$ holds $a_0|z|^2 \le z \cdot A(x)z \le A_0|z|^2$. Consider the second order elliptic operator Δ_A defined for each $\varphi \in C^2(\mathbb{R}^d)$ by

$$\Delta_A \varphi(x) := \operatorname{tr}(A(x) \operatorname{Hess} \varphi(x))$$

Due to results of [5], the following family $(F^A(t))_{t\ge 0}$ is Chernoff equivalent to the semigroup $(T_t)_{t\ge 0}$ on the space $C_\infty(\mathbb{R}^d) \equiv C_0(\mathbb{R}^d)$, generated by the closure of the operator $(L, C_c^{2,\alpha}(\mathbb{R}^d))$: $F^A(0) := \mathrm{Id}$ and for all $t > 0$ with $\varphi \in X$ and $x \in \mathbb{R}^d$

$$F^A(t)\varphi(x) := \frac{1}{\sqrt{(4\pi t)^d \det A(x)}} \int_{\mathbb{R}^d} \exp\left(-\frac{A^{-1}(x)(x-y)\cdot(x-y)}{4t} \right) \varphi(y)dy.$$

Combining this result with Theorem 2.1, one succeeds to weaken the assumption $a_0|z|^2 \le z \cdot A(x)z$ for all $x, z \in \mathbb{R}^d$ and some $a_0 > 0$:

Proposition 2.4. *Let A be as before. Let $a \in C_b(\mathbb{R}^d)$ be a scalar function with $a(x) > 0$ for all $x \in \mathbb{R}^d$. Consider $\widehat{A} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ such that $\widehat{A}(x) = a(x)A(x)$. Consider also the operator $\Delta_{\widehat{A}}$ defined for each $\varphi \in C^2(\mathbb{R}^d)$ by*

$$\Delta_{\widehat{A}}\varphi(x) := \operatorname{tr}(\widehat{A}(x) \operatorname{Hess} \varphi(x)) = a(x)\Delta_A\varphi(x) \equiv \widehat{\Delta_A}\varphi(x).$$

Assume the existence and strong continuity of the semigroup $(T_t^{\Delta_{\widehat{A}}})_{t\ge 0}$ generated by the closure of $(\Delta_{\widehat{A}}, C_c^{2,\alpha}(\mathbb{R}^d))$ on $X = C_\infty(\mathbb{R}^d)$. Consider the family $(\widehat{F}^A(t))_{t\ge 0}$ of linear operators on X defined by $\widehat{F}^A(0) := \mathrm{Id}$ and

$$\widehat{F}^A(t)\varphi(x) := \frac{1}{\sqrt{(4\pi t)^d \det \widehat{A}(x)}} \int_{\mathbb{R}^d} \exp\left(-\frac{\widehat{A}^{-1}(x)(x-y)\cdot(x-y)}{4t} \right) \varphi(y)dy$$

for all $t > 0$, $\varphi \in X$ and $x \in \mathbb{R}^d$. Then the family $(\widehat{F}^A(t))_{t\ge 0}$ is strongly continuous and Chernoff equivalent to the semigroup $(T_t^{\Delta_{\widehat{A}}})_{t\ge 0}$ on $C_\infty(\mathbb{R}^d)$ generated by the closure of $(\Delta_{\widehat{A}}, C_c^{2,\alpha}(\mathbb{R}^d))$. Therefore, the following Feynman Formula holds for all $\varphi \in X$, all $t > 0$ and all $x_0 \in \mathbb{R}^d$:

$$T_t^{\Delta_{\widehat{A}}}\varphi(x_0) = \\ = \lim_{n\to\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \varphi(x_n) p_{\widehat{A}}(t/n, x_0, x_1) \cdots p_{\widehat{A}}(t/n, x_{n-1}, x_n) dx_1 \dots dx_n,$$

where

$$p_{\widehat{A}}(t,x,y) := \frac{1}{\sqrt{(4\pi t a(x))^d \det A(x)}} \exp\left(- \frac{A^{-1}(x)(x-y)\cdot(x-y)}{4ta(x)} \right).$$

And the convergence is uniform with respect to $q_0 \in Q$ *and with respect to* $t \in (0, t^*]$ *for all* $t^* > 0$.

3. Conclusions

The method of Chernoff approximation can be applied to a wide class of Markov processes. In particular, it can be applied to processes obtained via a random time change from some original processes, whose transition kernels are known or already Chernoff-approximated. The constructed Chernoff approximations can be used for simulation of the considered time-changed processes.

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