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The Particles Population Propagation in Catalytic Branching Random Walk

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Abstract. For a supercritical catalytic branching random walk on $\mathbb{Z}^d$, $d \in \mathbb{N}$, with an arbitrary finite catalysts set we study the spread of particles population as time grows to infinity. Namely, we divide by t the position coordinates of each particle existing at time t and then let t tend to infinity. It is shown that in the limit there are a.s. no particles outside the closed convex surface in $\mathbb{R}^d$ which we call the propagation front and, under condition of infinite number of visits of the catalysts set, a.s. there exist particles on the propagation front. Recent strong limit theorems for total and local particles numbers established by the author play an essential role. The results obtained develop ones by Ph.Carmona and Y.Hu (2014) devoted to the spread of catalytic branching random walk on $\mathbb{Z}$.

Keywords: branching random walk, supercritical regime, spread of population, propagation front, many-to-one lemma.

1. Main section

We consider a catalytic branching random walk (CBRW) on $\mathbb{Z}^d$, $d \in \mathbb{N}$, with catalysts located at an arbitrary finite set $W \subset \mathbb{Z}^d$. Outside the set W a particle performs an ordinary random walk (in continuous time) generated by matrix $A = (a(x,y))_{x,y\in\mathbb{Z}^d}$. Being at a catalyst particles may either leave it or produce offspring according to a probability law depending on the catalyst. We have shown (see [1]) that combination of different characteristics of the random walk and branching at the catalysts set may lead to supercritical, critical or subcritical behavior of particles population. In particular, only in the supercritical regime with positive probability the total and local particles numbers jointly grow exponentially with some rate $\nu > 0$ (called the Malthusian parameter) as time tends to infinity. Thus in this case it is interesting to study the asymptotic shape of particles population propagation.

To establish almost sure results we assume that conditions in [2] describing the homogeneous in space and time CBRW hold (with $a(x,y) = a(0, y-x)$, $x, y \in \mathbb{Z}^d$) and also the Cramér condition is satisfied, i.e.

$$H(u) := \sum\nolimits_{x\in\mathbb{Z}^d} e^{\langle u,x\rangle} a(0,x) < \infty, \quad u \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^d$. Introduce the sets $R := \{u \in \mathbb{R}^d : H(u) = \nu\}$,

$$O_\varepsilon := \{x \in \mathbb{R}^d : \langle x, r \rangle > \nu + \varepsilon \text{ for at least one } r \in R\}, \quad \varepsilon \geq 0,$$

$$Q_\varepsilon := \{x \in \mathbb{R}^d : \langle x, r \rangle < \nu - \varepsilon \text{ for any } r \in R\}, \quad \varepsilon \in [0, \nu).$$

Let $B := \partial Q_0 (= \partial O_0)$ and $N(t) \subset \mathbb{Z}^d$ be the (random) set of particles existing in CBRW at time $t \geq 0$. For a particle $v \in N(t)$, denote by $X_v(t)$ its position in $\mathbb{Z}^d$ at time t. Introduce also the random set $I = \{\omega : \limsup_{t \to \infty} \{v \in N(t) : X_v(t) \in W\} \neq \varnothing\}$. The supercritical regime of CBRW guarantees that $\mathsf{P}_x(I) > 0$ where index $x \in \mathbb{Z}^d$ denotes the starting point of CBRW.

The following new result of [2] extends the corresponding one from [3].

Theorem 1 *Let a supercritical CBRW with Malthusian parameter $\nu > 0$ satisfy the above mentioned conditions. Then, for any $x \in \mathbb{Z}^d$ and $t \to \infty$, one has*

$$\mathsf{P}_x(\omega : \forall \varepsilon > 0 \; \exists t_0 = t_0(\omega, \varepsilon) \text{ s.t. } \forall t \geq t_0 \text{ and } \forall v \in N(t),\; X_v(t)/t \notin O_\varepsilon) = 1,$$

$$\mathsf{P}_x(\omega : \forall \varepsilon \in (0, \nu) \; \exists t_1 = t_1(\omega, \varepsilon) \text{ s.t. } \forall t \geq t_1 \; \exists v \in N(t),\; X_v(t)/t \notin Q_\varepsilon \,|\, I) = 1.$$

This Theorem means that if we divide the position coordinates of each particle existing in CBRW at time t by t and then let t tend to infinity, then in the limit there are a.s. no particles outside the set $B \cup Q_0$ and under condition of infinite number of visits of catalysts there are a.s. particles on B. In this sense it is natural to call the border B the *propagation front* of the particles population. Various examples are considered as well.

Mention in passing that related models and characteristics are also studied in [4] and [5]. However, in contrast to our results, the authors of those papers do not deal with the a.s. convergence.

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