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From the Pseudo-Poisson Processes with the Random Intensity to the Fractional Brownian Motion

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Abstract. We consider a Pseudo-Poisson process, when the leading Poisson process has a random intensity. Under an appropriate distribution for the random intensity the corresponding Pseudo-Poisson process possesses a covariance of the fractional Ornstein-Uhlenbeck process. Applying to the Pseudo-Poisson processes with the considered random intensity the Lamperti transform and then the Central Limit Theorem for vectors we obtain the fractional Brownian motion as a limit in a sense of weak convergence of finite dimensional distributions.

Keywords: Pseudo-Poisson process, Laplace transform, Lamperti transform, fractional Ornstein-Uhlenbeck process, fractional Brownian Motion .

1. Introduction

Pseudo-Poisson processes are introduced in the famous monograph of W.Feller, chapter X [1]. As defined in this monograph Pseudo-Poisson process represents a continuous random change (the Poisson randomization) of the "mathematical" time of a Markov sequence. So, Pseudo-Poisson process is a type of subordinators for random sequences, when the leading process is a Poisson one. We consider a generalization of the leading Poisson processes to the case of random intensity. Thus our subordination is driven by a Double Stochastic Poisson process, and such kind of subordinator for sequences we call as "Double Stochastic Pseudo-Poisson Process" or "Double Stochastic Poisson Stochastic Index process" (DS PSI process).

Key lemma states that in the case when the subordinated sequence consists of i.i.d. random variables with a finite variance the corresponding Double Stochastic Pseudo-Poisson process has a property of stationarity in the both wide and strict senses, and the covariance function is the Laplace transform of the random intensity. For a special type of discrete distribution for the random intensity the corresponding covariance function of the Double Stochastic Pseudo-Poisson process coincides with the covariance function for the Gaussian short-memory fractional Ornstein-Uhlenbeck (fO-U) process (on definition and main properties of f-OU we refer to [2]), i.e. for the Gaussian fO-U process with the Hurst parameter $0 < H < 1/2$. The well known fact is that the Lamperti transform [3] for the fO-U process allows us to obtain the fractional Brownian motion (fBm) process of the same Hurst parameter. Taking the driving Double Stochastic Poisson process with the such kind of type of distribution for the random intensity and applying this driving process to the subordination of the “mathematical” time of the random sequence with the terms having an arbitrary distribution of “the 2nd order” we obtain the stationary in a wide sense process with the same covariance as for the fO-U process with $0 < H < 1/2$. Next, applying the Lamperti transform to the corresponding Double Stochastic Pseudo-Poisson process we obtain a self-similar process with the stationary in a wide sense increments, with coefficient of self-similarity $0 < H < 1/2$, and with the truncations having the appropriate scaling distribution of the term of the subordinated sequence. Finally, under summation of independent copies of the Lamperti transformed DS PSI processes, normalized with $1/\sqrt{N}$, as a result of CLT we obtain as a limit (in a sense of the weak convergence of finite dimensional distributions) the Gaussian fractional Brownian motion process (fBm). Note, that summation of independent copies, normalized with $1/\sqrt{N}$, of the DS PSI processes without Lamperti transformations results in a limit fO-U process in a sense of so called “Upstairs representation” [4].

2. Main section

Let $\{\Omega, \mathcal{F}, \mathbf{P}\}$ be a probability space, $\omega \in \Omega$. Let $(\xi) = (\xi_j)$, $j = 0, 1, \ldots,$ be a random sequence, $\Pi_1(s)$, $s \in \mathbb{R}_+$ be the “standardized” Poisson process with the intensity 1, $\lambda(\omega)$ be a strictly positive random variable with the distribution function $F_{\lambda(\omega)}(x)$, $x > 0$. Assume that $\{(\xi), \Pi_1, \lambda(\omega)\}$ are jointly independent. We use the notation $\lambda(\omega)$ to differ the case of a random intensity from the case $\lambda = const$.

Definition 1. We define a Poisson subordinator with a random intensity $\lambda(\omega)$ for the sequence (ξ), or Double Stochastic Pseudo-Poisson Processes with a random intensity $\lambda(\omega)$ which is drawn at the initial time by the following change of time of (ξ)

$$\psi_{\lambda(\omega)} = \psi_{\lambda(\omega)}(s) = \psi(s;\, \lambda(\omega)) \stackrel{\triangle}{=} \xi_{\Pi_{\lambda(\omega)}(s)}, \qquad s \in \mathbb{R}_+\,,$$

where

$$\Pi_{\lambda(\omega)}(s) \stackrel{\triangle}{=} \Pi_1(s\lambda(\omega))\,.$$

In this paper we consider (ξ) consisting of i.i.d. random variables, $\mathbf{E}\xi_0 = 0$, $\mathbf{D}\xi_0 = 1$. The following key lemma states that the process $\psi_{\lambda(\omega)}$ is a strictly stationary one, and its covariance is the Laplace transform of $\lambda(\omega)$.

Lemma 1. The subordinator $\psi_{\lambda(\omega)}(t),\ \ t\in\mathbb{R}_+,$ is a strictly stationary process;

$$\operatorname{cov}\left(\psi_{\lambda(\omega)}(v),\psi_{\lambda(\omega)}(s+v)\right)=\int\limits_0^\infty \exp\{-ys\}\mathrm{d}F_{\lambda(\omega)}(y)\,,\quad \forall s,v\in\mathbb{R}_+\,.$$

For details and accuracy of Lemma 1 proof we refer to [5].

Note that in the case when the random intensity $\lambda(\omega)$ is not drawn at the initial time, but it follows the starting at zero positive Lévy process $\Lambda(t),\ \ t\geq 0$,

$$\operatorname{cov}(\psi(s),\psi(t))=L_{\Lambda(t-s)}(1),\quad t>s\,,$$

where $L_{\Lambda(t)}(u),\ \ u\geq 0$, is a Laplace transform defined on $\mathbb{R}_+\ni u$ for a truncation of the random process Λ at the point $t>0$.

According to the Barndorff-Nielsen's representation [2], we define a fractional Ornstein-Uhlenbeck process (fO-U) $U_H(t),\ t\in\mathbb{R},$ of the Hurst parameter $H\in(0,1]$, as a Gaussian centered stationary process with the covariance

$$r(t)=r_H(t)=\operatorname{cov}(U_H(0),U_H(t))$$
$$=\frac{1}{2}\left\{e^{-Ht}+e^{Ht}-|e^{t/2}-e^{-t/2}|^{2H}\right\}\,,\qquad t\in\mathbb{R}\,.$$

Remark that $r(t)=r(-t)$.

Due to Lamperti transform [3] the process $w_H(s)\triangleq s^HU_H(\log s)$, $s>0,\ w_H(0)=0$ a.s., is a fractional Brownian motion (fBm) process: Gaussian centered self-similar strictly stationary increments process with the Hurst parameter $H\in(0,1]$,

$$\operatorname{cov}(w_H(s),w_H(t))=\frac{1}{2}\left(s^{2H}+t^{2H}-|t-s|^{2H}\right)\,,\quad s,t\geq 0\,.$$

For $t\geq 0$ the following chain of simple equalities allows us to obtain the expression for $r(t)$ in a form of the Laplace transform of a probability distribution,

$$\begin{aligned} r(t)&=\frac{1}{2}e^{Ht}\left(1+e^{-2Ht}-\left(1-e^{-t}\right)^{2H}\right)\\ &=\frac{1}{2}e^{-Ht}+\frac{1}{2}e^{Ht}\left(1-\left(1-e^{-t}\right)^{2H}\right)\\ &=\frac{1}{2}e^{-Ht}+\frac{1}{2}\sum_{j=1}^{\infty}(-1)^{j-1}\binom{2H}{j}e^{-(j-H)t}\,, \end{aligned}\tag{1}$$

where

$$\binom{2H}{j}=\frac{\Gamma(2H+1)}{\Gamma(j+1)\Gamma(2H-j+1)}\,.$$

It is not difficult to check that in the case $0 < H < 1/2$ the expression (1) defines a fully monotone function equaling 1 at zero, hence it is the Laplace transform of a some probability distribution. The following random variable η_H (for $0 < H < 1/2$) possesses this distribution of a discrete type

$$\mathbf{P}(\eta_H = H) = \frac{1}{2} = p_0, \qquad \mathbf{P}(\eta_H = j - H) = \frac{1}{2} p_j, \quad j \in \mathbb{N};$$

$$p_1 = 2H, \qquad p_2 = \frac{2H(1-2H)}{2!}, \quad p_{k+1} = \left(1 - \frac{1+2H}{k+1}\right) p_k, \quad k \geq 2.$$

The random intensity $\lambda(\omega) \stackrel{d}{=} \eta_H$, substituted in Definition 1 of the DS PSI processes, (thanks to Lemma 1) provides that $\psi_H(s) = \xi_{\Pi_1(s\eta_H)}$, $s \in \mathbb{R}_+$, has the same covariance as for fO-U. Note that a pairwise characteristics function for the DS PSI process is as follows

$$L_{\lambda(\omega)}(s)\phi(\nu + \mu) + \left(1 - L_{\lambda(\omega)}(s)\right)\phi(\nu)\phi(\mu), \qquad \mu, \nu \in \mathbb{R}, \quad s \in \mathbb{R}_+.$$

Theorem 1. Let us extend the stationary $\psi_H(t)$ on $\mathbb{R} \ni t$ and consider independent copies $\psi_H^{[j]}(t)$, $j \in \mathbb{N}$, of $\psi_H(t)$ which subordinate (respectively) independent sequences $(\xi)^{[j]}$ of totally i.i.d random terms $\{\xi_i^{[j]}\}$, $i \in \mathbb{Z}_+$, $j \in \mathbb{N}$, with $\mathbb{E}\xi_0^{[1]} = 0$, $\mathbb{D}\xi_0^{[1]} = 1$.

Then the following convergence in a sense of convergence of finite dimensional distributions takes place as $N \to \infty$,

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \psi_H^{[j]}(t) \Rightarrow U_H(t), \quad t \in \mathbb{R};$$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} s^H \psi_H^{[j]}(\log s) \Rightarrow w_H(s), \quad s \in \mathbb{R}_+.$$

Proof of Theorem 1 directly follows from the Central Limit Theorem for vectors with the identical covariance.

For analysis of the distribution η_H let us introduce the following random variable η, taking values on $\{0, 1, 2, \ldots\}$,

$$\begin{aligned}
\mathbf{P}(\eta = 0) &= 2H, \\
\mathbf{P}(\eta = 1) &= \frac{2H(1-2H)}{2!}, \\
\ldots &= \ldots, \\
\mathbf{P}(\eta = k) &= \frac{2H(1-2H)(2-2H)\ldots(k-2H)}{(k+1)!}, \\
\ldots &= \ldots .
\end{aligned}$$

Obviously, the distribution of $\eta + (1 - H)$ equals (in law) to a conditional distribution of η_H provided that $(\eta_H \neq H)$.

After not difficult calculations we obtain a distribution function for η in the following explicit form

$$\mathbf{P}(\eta \leq n) = 1 - \binom{n+1}{n+1-2H}, \qquad n \in \mathbb{N},$$

and the asymptotic of its tail

$$\mathbf{P}(\eta \geq n) \sim \frac{n^{-2H}}{\Gamma(1-2H)}, \qquad n \to \infty.$$

3. Conclusions

Limit expressions for U_H and w_H in Theorem 1 and explicit representation of the distribution of η_H set an algorithm to modelling as well fO-U, as fBm.

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