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Markovian Modelling of Arrival Processes

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Abstract. Markovian arrival process is a popular tool for modeling arrival processes of stochastic systems such as queueing systems, reliability systems and telecommunications networks. In this paper we describe the development of Markovian arrival processes before 1990, the year which is considered by the western research community to be the year of its introduction.

Keywords: MC-stream, Markovian arrival process.

1. Introduction

Markovian arrival process (MAP) is a popular tool for modeling arrival processes of stochastic systems. MAP is generalization of point processes generated by transitions of Markov chains [1, 2]. Its properties can be derived from the theory of two-dimensional Markov processes with homogeneous second component, developed in [3]. Properties of MAP were studied in [4–12] under the name "arrival stream without after-effect driven by a Markov chain" or simply "MC-stream". In these papers a matrix representation for MC-stream was introduced, which made it easy to use it in stochastic modeling. In parallel, starting from the paper [13] MAP was also studied under the name "versatile Markovian point process". Finally, its name was settled at Markovian arrival processes in [14], as well as matrix representation used for MC-stream become common. In this paper we show the development of the theory of Markovian Arrival Processes in RUDN University before 1990.

2. General Markovian arrival process

Consider a multi-type arrival process $\mathcal{T} = ((\tau_n, c_n))$, where $0 \leq \tau_1 \leq \tau_2 \leq ...$ are arrival times and $\sigma_n \in \{1, 2, ..., K\}$ is the type of arrival at time τ_n. Let $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise, $N_k(t) = \sum_{\tau_n \leq t} \delta_{k,\tau_n}$ be the counting process of type k arrivals and $\mathbf{N}(t) = (N_1(t), N_2(t), ..., N_K(t))$. We say that a point process $\mathcal{T}$ is the (multi-type) Markovian arrival process if there exists random process $X(t)$ with a finite state space $\mathcal{X} = \{1, 2, ..., L\}$ such that $Y(t) = (X(t), N(t))$ is the time-homogeneous Markov process with homogeneous second component [3], i.e. if for all $\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}$,$i, j \in \mathcal{X}$ and $s, t \geq 0$ we have

$$\mathrm{P}\{X(s+t)=j,\mathbf{N}(s+t)=\mathbf{n}|X(s)=i,\mathbf{N}(s)=\mathbf{k}\}=p_{i,j}(\mathbf{n}-\mathbf{k},t).$$

The following properties of MAP follow from general results in [3].

The phase process $X(t)$ is the time-homogeneous Markov chain with the matrix of transition probabilities

$$\mathbf{P}(t)=\sum_{\mathbf{n}\geq\mathbf{0}}\mathbf{P}(\mathbf{n},t),$$

where $\mathbf{P}(\mathbf{n},t)=[p_{i,j}(\mathbf{n},t)]$. Consider matrices of transition rates $\mathbf{A}(\mathbf{n})=[a_{i,j}(\mathbf{n})]$ of the process $Y(t)$,

$$a_{i,i}(\mathbf{0})=\frac{1}{\delta}\lim_{\delta\to 0}(p_{i,i}(\mathbf{0},\delta)-1),a_{i,j}(\mathbf{n})=\frac{1}{\delta}\lim_{\delta\to 0}p_{i,j}(\mathbf{n},\delta),i\neq j,\mathbf{n}\geq\mathbf{0}.$$

Matrices $\mathbf{A}(\mathbf{n}),\mathbf{n}\neq\mathbf{0}$, are nonnegative, $\mathbf{A}(\mathbf{0})$ has nonnegative off-diagonal elements, and the matrix $\mathbf{A}=[a_{i,j}]$ given by

$$\mathbf{A}=\sum_{\mathbf{n}\geq\mathbf{0}}\mathbf{A}(\mathbf{n})$$

is the generator matrix of the phase process $X(t)$. We assume that the generator $\mathbf{A}$ is irreducible and denote $\mathbf{p}=(p_i)$ the stationary probability vector of $X(t)$.

The counting process $\mathbf{N}(t)$ of Markovian arrival processes is the process with independent increments driven by a Markov chain [3]. In [15] arrival processes having counting process with independent increments were called "arrival streams without after-effect". For this reason in some papers Markovian arrival process was called "arrival stream without after-effect driven by a Markov chain" until in 1978 during seminar in Moscow State University G.P. Klimov proposed to call it shortly as "MC-stream".

3. Simple Markovian arrival process

Simple Markovian arrival process is a single-type MAP characterized by two nonzero rate matrices $\mathbf{S}=\mathbf{A}(0)$ and $\mathbf{R}=\mathbf{A}(1)$, while $\mathbf{A}(k)=\mathbf{0}$, for $k\geq 2$. One can find properties of the simple MAPs in [7]. Simple MAP can be interpreted as follows [8]. Let arrivals occur with probability φ_{ij} at each transition of the phase process $X(t)$from i to j and while $X(t)$is in the state i arrival process is the Poisson with the rate λ_i. Such an arrival process is a simple MAP with matrices $\mathbf{S}=[s_{i,j}]$ and $\mathbf{R}=[r_{i,j}]$given by

$$s_{i,j}=\begin{cases} a_{i,j}(1-\varphi_{i,j}), & i\neq j,\\ a_{i,i}-\lambda_i, & i=j,\end{cases}\quad r_{i,j}=\begin{cases} a_{i,j}\varphi_{i,j}, & i\neq j,\\ \lambda_i, & i=j.\end{cases}$$

Simple Markovian arrival processes have been used for the first time in the study a finite queueing system that can be described by a Markov chain $(X(t), Y(t))$, where $X(t)$ is the phase process of a simple MAP and the process $Y(t)$ describes internal system state [11].

Counting process $N(t)$of the stationary MAP is asymptotically Normal with mean $M(t) = \lambda t$ and variance

$$D(t) = (2\mathbf{q}_1\mathbf{R}\mathbf{u} - \lambda)t + 2(\mathbf{q}_2\mathbf{R}\mathbf{u} - \lambda) + o(1),$$

where $\lambda = \mathbf{pRu}$, $\mathbf{u}$ is the column vector of all ones, and row vectors $\mathbf{q}_1$and $\mathbf{q}_2$ are unique solutions of the linear systems (1)

$$\mathbf{q}_1\mathbf{A} = \mathbf{p}(\lambda\mathbf{I} - \mathbf{R}), \quad \mathbf{q}_1\mathbf{u} = 1, \mathbf{q}_2\mathbf{A} = \mathbf{q}_1 - \mathbf{p}, \quad \mathbf{q}_2\mathbf{u} = 1. \qquad (1)$$

In [10] multi-server queueing systems with MAP arrivals and exponentially distributed service times with parameter μ were considered. Binomial moments of the stationary probability distribution of the number of busy servers in the infinite-server MAP|M|$\propto$ system is given by

$$b_n = \mathbf{p}\mathbf{G}(\mu)\mathbf{G}(2\mu)\cdots\mathbf{G}(n\mu)\mathbf{u},$$

where $\mathbf{G}(s) = \mathbf{R}(s\mathbf{I} - \mathbf{A})^{-1}$. For the MAP with diagonal matrix $\mathbf{R}$ formula (1) was derived in [4]. Stationary loss probability for finite-server MAP|M|c|0 loss system is given by

$$B = \frac{1}{\lambda}\mathbf{gRu},$$

where the row vector $\mathbf{g}$ is the unique solution of the linear system

$$\mathbf{g}\left(\mathbf{I} + \sum_{k=1}^{c}\binom{n}{k-1}\mathbf{G}(k\mu)\mathbf{G}((k+1)\mu)\cdots\mathbf{G}(c\mu)\right) =$$
$$= \mathbf{p}\mathbf{G}(\mu)\mathbf{G}(2\mu)\cdots\mathbf{G}(c\mu).$$

If the matrix $\mathbf{R}$ has the rank one then the MAP is the renewal process of PH-type. If the vector $\mathbf{p}$ is the left or the vector $\mathbf{u}$ is the right eigenvector of $\mathbf{R}$ then the MAP is the Poisson process [10].

4. Multi-type Markovian arrival process

Simple multi-type Markovian arrival process is a generalization of simple single-type MAP . This version of MAP is characterized by $K + 1$ nonzero matrices $\mathbf{S} = \mathbf{A}(0)$ and $\mathbf{R}_k = \mathbf{A}(\mathbf{e}_k)$, $k = 1, 2, ..., K$, where vector $\mathbf{e}_k$ has its kth component equal to 1 and all other components equal to zero. Properties of the simple multi-type MAPs were studied in [11].

Recursive method for calculation loss probability for the system MAP|M|c|0 with multi-type arrival process was given in [12]. Matrices $\mathbf{\Psi}_k$ defined by the recursion

$$\mathbf{\Psi}_0 = \mathbf{0}, \mathbf{\Psi}_k = \left(\mathbf{I} - \frac{1}{k}(\mathbf{S} + \mathbf{\Psi}_{k-1}\mathbf{R})\right)^{-1}, \ k = 1, 2, \ldots$$

are nonnegative and satisfies the following inequalities

$$\mathbf{\Psi}_k \mathbf{R} \mathbf{u} \le k\mathbf{u}, \mathbf{p}\mathbf{\Psi}_k \le \mathbf{p}.$$

Stationary time blocking probability E and type k call blocking probability B_k

$$E = \mathbf{p}(\mathbf{I} - \mathbf{\Psi}_c \mathbf{u}, B_k = \frac{1}{\lambda_k}\mathbf{p}(\mathbf{I} - \mathbf{\Psi}_c)\mathbf{R}_k \mathbf{u},$$

where $\lambda_k = \mathbf{p}\mathbf{R}_k\mathbf{u}$ is the arrival rate of type k calls.

5. Conclusions

In this paper we show that the theory of Markovian Arrival Processes was developed in RUDN University long before 1990, the year when the paper [14] was published, and which is considered by the western research community to be the year of the MAP introduction.

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References

1. *Basharin G. P.* O vychislenii momentov obsluzhennojj i izbytochnojj nagruzok slozhnojj sistemy [On the calculation of the moments of the serviced and lost traffic of a complex system] // Izvestiya Akademii Nauk SSSR, Tekhnicheskaia Kibernetika. — 1972. — P. 42-51.
2. *Rudemo M.* Point processes generated by transitions of Markov chains // Advances in Applied Probability. — 1973. — Vol. 5, no. 2. — P. 262–286.
3. *Ezhov I. I., Skorokhod A. V.* Markov processes with homogeneous second component // Theory of Probability and its Applications. — 1969. — Part I: Vol. 14, no.1. — P. 1-13. — Part II: Vol. 14, no.4. — P. 652–667.
4. *Kokotushkin V. A.* Nekotorye zadachi matematicheskojj teorii teletrafika [Some mathematical problems of teletraffic theory]. PhD thesis. — Peoples Friendship University, Moscow, 1974.

5. *Naumov V.* O nezavisimojj rabote podsistem slozhnojj sistemy [On the independent behaviour of subsystems of complicated system] // Queuing Theory. Proc. 3d Seminar on Queueing Theory, Puschtschino, 1974 / B. V. Gnedenko, Yu. I. Gromak, E. V. Chepurin. — Moscow State University, 1976. — P. 169-177.
6. *Bocharov P. P., Naumov V. A.* Analiz giperehksponencial'nojj dvukhfaznojj sistemy s ogranichennym promezhutochnym nakopitelem [Analysis of hyperexponential tandem queue with finite intermediate room] // Informacionnye seti i ikh struktura. — M.: Nauka, 1976. — P. 168–181.
7. *Naumov V. A.* Issledovanie nekotorykh mnogofaznykh sistem massovogo obsluzhivanija [Analysis of some queues in series]. PhD thesis. — Peoples Friendship University, Moscow, 1978.
8. *Naumov V. A.* Ob obsluzhennojj i izbytochnojj nagruzkakh polnodostupnogo puchka s ogranichennojj ochered'ju [On the calculation of the moments of the serviced and overflow traffic of a full-available loss system with finite waiting room] // Chislennye metody reshenija zadach matematicheskojj fiziki i teorii sistem. — Peoples Friendship University, 1977. — P. 51–55.
9. *Basharin G. P., Kokotushkin V. A., Naumov V. A.* The method of equivalent substitution for calculating fragments of communication networks for a digital computer, Part I // Engineering Cybernetics. — 1979. — Vol. 17, no. 6. — P. 66–79. Translatied from: // Izvestiya Akademii Nauk SSSR. Tekhnicheskaia Kibernetika. — 1979. — No. 6. — P. 92–99.
10. *Basharin G. P., Naumov V. A.* Simple matrix description of peaked and smooth traffic and its applications // Fundamentals of Teletraffic Theory, Proc. 3d Int. Seminar on Teletraffic Theory — VINITI, 1984. — P. 38–44.
11. *Naumov V. A.* Markovskie modeli potokov trebovanijj [Markovian modeling of arrival processes] // Sistemy massovogo obsluzhivanija i informatika — Peoples Friendship University, 1987. — P. 67–73.
12. *Naumov V. A.* Matrichnyjj analog formulju Erlanga [Matrix analogy of Erlang formula] // Sb. trudov 10 Vsesojuznojj shkoly-seminara po teorii teletrafika "Modeli raspredelenija informacii i metody ikh analiza". — VINITI, Moscow, 1988. — P. 39–43.
13. *Neuts M. F.* A versatile Markovian point process // J. of Applied Probability. — 1979. — Vol. 16. — P. 764–779.
14. *Lucantoni D. M., Meier-Hellstern K., Neuts M. F.* A single-server queue with server vacations and a class of non-renewal arrival processes // Adv. in Applied Probability. — 1990. — Vol. 22. — P. 676-705.
15. *Khinchin A. Ya.* Sequences of Chance Events without After-Effects // Theory of Probability and its Applications. — 1956. — Vol. 1, no. 1. — P. 1–15.