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One server queue with bulk arrivals

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Abstract. This paper deals with a queuing system $GI^{\nu}|M|1|\infty$, *i.e.*, single server queue with general renewal arrivals, exponentially distributed service times and infinite number of waiting positions. The purpose is to find the steady-state results in terms of the probability-generating functions for the number of customers in the queue.

Keywords: queueing system, batch arrivals, probability generating functions, embedded Markov chain.

1. Introduction

Consider a queueing system $GI^{\nu}|M|1|\infty$. This means that customer arrival moments $0 < t_1 < t_2 < ... < t_n < ...$ constitute a renewal process [1] with the distribution function $P\{t_n - t_{n-1} < t\} = F(t)$.

At every moment t_n a group of ν_n customers arrives, with ν_n being independent and equally distributed. Additionally suppose that ν_n are bounded

$$\alpha(z) = Mz^{\nu_n} = \alpha_1 z + \alpha_2 z^2 + ... + \alpha_m z^m \,,\; \alpha_m \neq 0\,.$$

The system has single service channel and service time is exponentially distributed with parameter μ. Waiting queue size is unlimited and customers are serviced in the order of their arrival.

Let $\xi(t)$ be the number of customers in the queueing system at any given moment t. Our aim is to find stationary distribution of the process

$$P(z) = \lim_{t\to\infty} Mz^{\xi(t)} = \sum_{n=0}^{\infty} p_n z^n\,.$$

2. Analysis of the embedded queue

Consider process $\xi(t)$ at customer arrival moments and denote

$$\xi_n = \xi(t_n - 0),\; n = 1, 2, ... ,\; \xi_1 = 0\,.$$

Then it's obvious that the sequence of ξ_n constitutes a homogenous Markov chain. Let η_n be the number of points of a Poisson process with parameter μ falling in the interval (t_n, t_{n+1}). Random variables η_n are independent and their distribution is

$$Mz^{\eta_n} = \sum_{s=0}^{\infty} \omega_s z^s = \sum_{s=0}^{\infty} z^s \int_0^{\infty} \frac{(\mu x)^s}{s!} e^{-\mu x} dF(x) =$$
$$= \int_0^{\infty} e^{-\mu x(1-z)} dF(x) = \varphi(\mu - \mu z),$$

where $\varphi(s) = \int_0^{\infty} e^{-sx} dF(x)$, and $\omega_s = \int_0^{\infty} \frac{(\mu x)^s}{s!} e^{-\mu x} dF(x)$, $s \geqslant 0$, is the probability of having exactly s customers serviced in the interval between two consequent arrivals under the condition that after first of those arrivals total amount of customers in the system is greater than s, that is $P\{\eta_n = s\} = \omega_s$.

It is easy to see that the following equations holds between random variables

$$\xi_{n+1} = (\xi_n + \nu_n - \eta_n)_+ \ , \ n \geqslant 1 \ , \ x_+ = \frac{x + |x|}{2} \ . \tag{1}$$

It is known [1] that Markov chain defined by equation (1) has a stationary distribution if and only if $M(\nu_n - \eta_n) < 0$, or

$$\nu = \sum_{k=0}^{m} k\alpha_k = \alpha'(1) < \mu T \ , \tag{2}$$

where $T = \int_0^{\infty} x dF(x)$ is the average time between customer arrivals.

Supposing this inequality holds, let's find stationary distribution of the chain ξ_n. Let $\pi(z) = \lim_{t\to\infty} Mz^{\xi_n} = \sum_{k=0}^{\infty} \pi_k z^k$. From the recurrent equation (1) the following representation for function $\pi(z)$ could be obtained:

$$\pi(z) = \frac{Q\left(\frac{1}{z}\right)}{1 - \alpha(z)\varphi\left(\mu - \frac{\mu}{z}\right)} \ , \tag{3}$$

where $Q(z) = \sum_{k+l<s} \pi_k \alpha_l \omega_s \left(1 - z^{m-k-l}\right)$. This could be shown, for example, as follows.

For variables satisfying (1) the following equations hold:

$$\pi_0 = \sum_{k+l-s\leqslant 0} \pi_k \alpha_l \omega_s \ \text{ and } \ \pi_n = \sum_{k+l-s=n} \pi_k \alpha_l \omega_s \ , n > 0 \ .$$

If we multiply left and right side of each of these equations by appropriate power of z and add, we obtain:

$$\sum_{n\geqslant 0} \pi_n z^n = \sum_{n>0} \sum_{k+l-s=n} \pi_k \alpha_l \omega_s z^n + \sum_{k+l-s\leqslant 0} \pi_k \alpha_l \omega_s \ .$$

then with some more easy and obvious transformations we get

$$\sum_{n\geqslant 0}\pi_n z^n = \sum_{n>0}\sum_{k+l-s=n}\pi_k\alpha_l\omega_s z^{k+l-s} +$$
$$+\sum_{k+l-s\leqslant 0}\pi_k\alpha_l\omega_s\left(1-z^{k+l-s}+z^{k+l-s}\right)$$

and finally

$$\pi(z) = \sum_{k+l-s>0}\pi_k\alpha_l\omega_s z^{k+l-s} + \sum_{k+l-s\leqslant 0}\pi_k\alpha_l\omega_s z^{k+l-s} +$$
$$+\sum_{k+l-s\leqslant 0}\pi_k\alpha_l\omega_s\left(1-z^{k+l-s}\right).$$

If we now notice, that in the right side of the last equation first two sums are equal to $\pi(z)\alpha(z)\varphi\left(\mu-\frac{\mu}{z}\right)$, and in the third sum the expression $1-z^{k+l-s}$ equals zero for $k+l-s=0$, the validity of (3) becomes obvious. Let's now prove one auxiliary statement.

Lemma. *Equation* $\alpha(z)\varphi\left(\mu-\frac{\mu}{z}\right)=1$ *has, counted with multiplicity, exactly* m *roots for* $|z|>1$.

Proof. It is sufficient to prove that the equation $\alpha(\frac{1}{z})\varphi\left(\mu-\mu z\right)=1$ has exactly m roots inside the unit disk. Multiply both parts of the last equation by z^m. This won't add any new roots because of $\alpha_m\neq 0$. Now consider the equation $z^m - z^m\alpha(\frac{1}{z})\varphi\left(\mu-\mu z\right)=0$. Function $f(x)=x^m\alpha(\frac{1}{x})\varphi\left(\mu-\mu x\right)$ defined on $[0,1]$ has derivative $f'(x)=m-\nu+mT>m$ according to (2). On the other hand, $\left(z^m\right)'_{z=1}=m$. Thus, for $r\in(0,1)$ and sufficiently close to 1, $f(r)<r^m$. Taking $z=re^{it}$ we get $\left|z^m\alpha(\frac{1}{z})\varphi\left(\mu-\mu z\right)\right|\leqslant r^m\alpha(\frac{1}{r})\varphi\left(\mu-\mu r\right)<r^m=|z|^m$ and then the statement of the lemma follows from Rouche's theorem [2]. □

Denote as $\lambda_1,\lambda_2,...,\lambda_m$ roots of the equation $\alpha(\frac{1}{z})\varphi\left(\mu-\mu z\right)=1$. Then roots of the original equation are $\frac{1}{\lambda_1},\frac{1}{\lambda_2},...,\frac{1}{\lambda_m}$.

Theorem 1. *Under the condition* (2) *stationary distribution of the embedded chain exists and is defined by the generating function*

$$\pi(z)=\frac{(1-\lambda_1)(1-\lambda_2)...(1-\lambda_m)}{(1-\lambda_1 z)(1-\lambda_2 z)...(1-\lambda_m z)}. \tag{4}$$

Proof. Multiplying both parts of (3) by $(1-\lambda_1 z)...(1-\lambda_m z)$, we get

$$\Phi_1(z):=(1-\lambda_1 z)...(1-\lambda_m z)\,\pi(z)=$$

$$=\frac{Q\left(\frac{1}{z}\right)(1-\lambda_1 z)...(1-\lambda_m z)}{1-\alpha(z)\varphi\left(\mu-\frac{\mu}{z}\right)}=:\Phi_2(z).$$

Function $\Phi_1(z)$ is continous for $|z| \leqslant 1$ ($z = 1$ is a removable discontinuity) and analytic for $|z| < 1$. Function $\Phi_2(z)$ is continous for $|z| \geqslant 1$ ($z = 1$ is a removable discontinuity) and analytic for $|z| > 1$. Because $\lambda_k \neq 0$ and $\lambda_m \neq 0$, $\Phi_2(\infty) = C$.

Thus [2] function $\Phi(z) = \begin{cases} \Phi_1(z), & |z| \leqslant 1 \\ \Phi_2(z), & |z| > 1 \end{cases}$ is analytic and bounded on the whole plane. Then by Liouville's theorem [2] it is constant, *i.e.*, $(1-\lambda_1 z)\ldots(1-\lambda_m z)\,\pi(z) = C$. Using normalization condition $\pi(1) = 1$, we get the expression (4). $\square$

3. Stationary distribution of the process $\xi(t)$

Consider embedded in process $\xi(t)$ semi-Markov process [3] $\zeta(t)$, defined by conditions $\zeta(t) = \xi(t_n - 0)$ for $t \in [t_n, t_{n+1})$. From the theory of semi-Markov processes [3] it follows that the stationary transition intensity of process$\zeta(t)$ to state k equals $h_k = \frac{\pi_k}{T}$. It also follows from the same theory that stationary distribution of $\xi(t)$ exists. Stationary probabilities of the process $\xi(t)$ are expressed as follows: for $n > 0$

$$p_n = \sum_{k+l \geqslant n} \tfrac{\pi_k}{T}\alpha_l \int_0^\infty \frac{(\mu x)^{k+l-n}}{(k+l-n)!} e^{-\mu x}\overline{F}(x)dx = \tfrac{1}{T}\sum_{k+l \geqslant n} \pi_k \alpha_l \omega_{k+l-n}\,,$$

$$\overline{F}(x) = 1 - F(x),\ \omega_s = \int_0^\infty \tfrac{(\mu x)^s}{(s)!} e^{-\mu x}\overline{F}(x)dx\,.$$

Verbally this expression could be derived this way: for the process in stationary mode at a given moment of time to be in a state n it is necessary, that at some earlier moment at a distance x semi-Markov process $\zeta(t)$ gets to some state k with probability $h_k dx$, at that moment a new group of l customers arrives, and then through the time x no new groups arrive, probability of this being $\overline{F}(x)$, and during the time x exactly $k + l - n$ customers were served, probability of this being $\frac{(\mu x)^{k+l-n}}{(k+l-n)!} e^{-\mu x}$. Adding this up by k and l and integrating over x, we obtain the expression above. In a similar way we get probability $p_0 = \frac{1}{T} \sum\limits_{k+l \leqslant s} \pi_k \alpha_l \omega_s$.

With transformations similar to those used above to obtain (3) we get this equation for the generating function of the stationary probabilities:

$$P(z) = \sum_{n=0}^{\infty} p_n z^n = \tfrac{1}{T}\sum_{k,l,s} \pi_k \alpha_l \omega_s z^{k+l-s} + \tfrac{1}{T}\sum_{k+l-s<0} \pi_k \alpha_l \omega_s \left(1 - z^{k+l-s}\right)\,,$$

where

$$\sum_{s=0}^{\infty}\omega_s z^{-s} = \int_0^\infty e^{-\mu x\left(1-\frac{1}{z}\right)}\overline{F}(x)dx = \frac{1}{\mu}\frac{z}{1-z}\int_0^\infty \overline{F}(x)de^{-\mu x\left(1-\frac{1}{z}\right)} =$$
$$= \frac{1}{\mu}\frac{z}{1-z}\left(-1+\varphi\left(\mu-\frac{\mu}{z}\right)\right).$$

Now generating function of the stationary probabilities could be easily expressed as

$$P(z) = \frac{1}{\mu T}\pi(z)\alpha(z)\frac{z\left(1-\varphi\left(\mu-\frac{\mu}{z}\right)\right)}{z-1} + Q_1\left(\frac{1}{z}\right),$$

where $Q_1(z) = \frac{1}{T}\sum\limits_{k+l<s}\pi_k\alpha_l\omega_s\left(1-z^{s-k-l}\right)$.

Expressing from (3) the product $\alpha(z)\pi(z)\varphi\left(\mu-\frac{\mu}{z}\right)$ in terms of $\pi(z)$ we obtain

$$P(z) = \frac{1}{\mu T}z\pi(z)\frac{1-\alpha(z)}{1-z} + Q_2\left(\frac{1}{z}\right),$$

where $Q_2\left(\frac{1}{z}\right) = Q_1\left(\frac{1}{z}\right) + \frac{1}{\mu T}\frac{z}{z-1}Q\left(\frac{1}{z}\right)$.

Similar to the proof of theorem 1, we can now deduce, that function $P(z) - \frac{1}{\mu T}z\pi(z)\frac{1-\alpha(z)}{1-z}$ is analytic and bounded on the whole plane, therefore being constant, its value obtained from the normalization condition $P(1) = 1$. Therefore we obtain final result.

Theorem 2. *Under the condition* (2) *stationary distribution of the process* $\xi(t)$ *exists and is defined by the generating function*

$$P(z) = \sum_{n=0}^{\infty}p_n z^n = \frac{1}{\mu T}z\pi(z)\frac{1-\alpha(z)}{1-z} + 1 - \frac{\nu}{\mu T},$$

where $\pi(z)$ *is defined by* (4).

References

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