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On natural convergence rate estimates in the Lindeberg theorem

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Abstract. We present various natural (according to the Zolotarev (1986) classification) convergence rate estimates in the Lindeberg theorem, like Esseen (1969), Rozovsky (1974), and Ahmad-Wang (2016) inequalities, and their improvements. We also discuss some problems associated with the optimization of the appearing absolute constants and present both upper and lower bounds for them.

Keywords: analytical methods in probability theory, computational methods, central limit theorem, characteristic function.

Let $X_1, X_2, \ldots, X_n$ be independent r.v.'s with distribution functions $F_1, \ldots, F_n$ and such that $\mathbb{E}X_k = 0$, $\mathbb{E}X_k^2 = \sigma_k^2 < \infty$. Let $B_n^2 = \sum_1^n \sigma_k^2 > 0$, $S_n = B_n^{-1}\sum_{k=1}^n X_k$, $\overline{F}_n(x) = \mathsf{P}(S_n < x)$, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}dt$,

$$\Delta_n = \sup_x \left|\overline{F}_n(x) - \Phi(x)\right|, \; \varrho_k(z) = \left|\int_{|x|<z} x^3\, dF_k(x)\right| + z\int_{|x|\geqslant z} x^2\, dF_k(x),$$

$$L_{E,n}^3 = B_n^{-3}\sum_{k=1}^n \sup_{z>0}\{\varrho_k(z)\}, \quad \overline{L}_{E,n}^3 = B_n^{-3}\sum_{k=1}^n \sup_{0<z\leqslant B_n}\{\varrho_k(z)\},$$

$$L_{R,n}^3 = B_n^{-3}\left[\left|\sum_{k=1}^n\int_{|x|<B_n} x^3\, dF_k(x)\right| + \sup_{0<z\leqslant B_n}\left\{z\sum_{k=1}^n\int_{|x|\geqslant z} x^2\, dF_k(x)\right\}\right].$$

The first term in $\varrho_k(z)$ is the *truncated third-order moment* and the second one is the *quadratic tail.* Note that $\overline{L}_{E,n}^3, L_{R,n}^3$ are always finite, while $L_{E,n}^3$ maybe infinite. Moreover, $\overline{L}_{E,n}^3 \leqslant L_{E,n}^3$.

In (Esseen, 1968) the following estimates are given:

$$\Delta_n \leqslant C_E \cdot L_{E,n}^3 \quad \text{and} \quad \Delta_n \leqslant \overline{C}_E \cdot \overline{L}_{E,n}^3, \tag{1}$$

where C_E, $\overline{C}_E$ are some absolute constants. In (Rozovskii, 1974) it was proved that

$$\Delta_n \leqslant C_R \cdot L_{R,n}^3, \tag{2}$$

where C_R is an absolute constant.

A very recent result due to (Ahmad, Wang, 2016) can be formulated in the following terms. Let $\mathcal{G}$ be a class of all functions $g\colon [0,\infty)\to[0,\infty)$ such that $g(z)$ does not decrease together with $z/g(z)$ for $z>0$, then for every $g\in\mathcal{G}$

$$\Delta_n \leqslant \frac{C_{AW}}{B_n^2 g(B_n)} \sum_{k=1}^{n} \sup_{z>0} \frac{g(z)}{z} \varrho_k(z), \tag{3}$$

where C_{AW} is an absolute constant.

All these inequalities improve the celebrated Osipov (1966), Katz (1963), and Petrov (1965) estimates in the CLT and thus can be called *natural* (according to Zolotarev's (1986) classification) convergence rate estimates in the CLT in the sense that their left- and right-hand sides either tend or do not tend to zero simultaneously, if the X_k satisfy the Feller condition of the uniform asymptotic negligibility.

In the present work we improve all the bounds presented above by:

(i) transferring the suprema and the modulae outside of the sum over $k=\overline{1,n}$ in (1) and (3);

(ii) bounding the domain $z>0$ over which the supremum can be taken in (3);

(iii) introducing a parameter $\varepsilon\in(0,\infty]$ defining the domain $0<z<\varepsilon B_n$ over which the suprema can be taken in (1)–(3);

(iv) introducing a parameter $\gamma>0$ defining the proportion between the third truncated moment and the quadratic tail in the definitions of $\varrho_k(z)$ and $L^3_{R,n}$.

We also find the extremal functions g that minimize and maximize the R.H.S. of inequality (3) and its improvements. Moreover, we present two–sided bounds for all the constants appearing in the improved inequalities, the lower bounds being obtained in terms of the asymptotically exact constants introduced in (Komogorov, 1953), (Zolotarev, 1986), and (Shevtsova, 2011). The proof of the upper bounds is substantially based on the method of characteristic functions, for which purpose new and sharp estimates for characteristic function were obtained.

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