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On probabilities of large and moderate deviations for L-statistics: a survey of some recent developments

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Abstract. A survey of some new results on Cramér type large and moderate deviations for trimmed L-statistics and for intermediate trimmed means will be presented. We discuss a new approach to the investigation of asymptotic properties of trimmed L-statistics proposed in the reviewed papers that allowed us to establish certain results on large and moderate deviations under quite mild and natural conditions.

Keywords: large deviations, moderate deviations, L-statistics, intermediate trimmed mean, slightly trimmed mean, central limit theorem for L-statistics.

1. Introduction

The class of L-statistics is one of the most commonly used classes in statistical inferences. We refer to monographs [5], [11], [12], [14] for the introduction to the theory of L-statistics. A survey on some modern applications of them in the economy and theory of actuarial risks can be found in [7]. There is an extensive literature on asymptotic properties of L-statistics, but its part concerning the large deviations is not so vast. We can mention a few of highly sharp results on this topic for L-statistics with smooth weight functions established in [13], [2], [1]. As to the trimmed L-statistics, the first – and up to the recent time the single – result on probabilities of large deviations was obtained in [4], but under some strict and unnatural conditions. Recently, the latter result was strengthened in [9], where a different approach than in [4] was proposed and implemented.

In this note we present some of our recent results established in [8]- [10].

To conclude this short introduction, we want to mention a paper [3], and an interesting article [6], in which a general delta method in the theory of Chernoff's type large deviations was proposed and illustrated by many examples including M-estimators and L-statistics.

2. Moderate deviations for intermediate trimmed means

Let $X_1, X_2, \dots$ be a sequence of independent identically distributed (i.i.d.) real-valued random variables (r.v.'s) with common distribution function (df) F, and for each integer $n \geq 1$ let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Introduce the left-continuous inverse function F^{-1} defined as $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $0 < u \leq 1$, $F^{-1}(0) = F^{-1}(0^+)$, and let F_n and F_n^{-1} denote the empirical *df* and its inverse respectively.

Consider the intermediate trimmed mean

$$T_n = \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i:n} = \int_{\alpha_n}^{1-\beta_n} F_n^{-1}(u)\, d\, u,$$

where k_n, m_n are two sequences of integers such that $0 \leq k_n < n - m_n \leq n$, $\alpha_n = k_n/n$, $\beta_n = m_n/n$, where we assume that

$$\min(k_n, m_n) \to \infty, \quad \max(\alpha_n, \beta_n) \to 0, \quad \text{as } n \to \infty.$$

Define the population trimmed mean

$$\mu(u, 1-v) = \int_u^{1-v} F^{-1}(s)\, d\, s, \quad \text{where} \quad 0 \leq u < 1 - v \leq 1.$$

Let $\xi_\nu = F^{-1}(\nu)$ denote the ν-th quantile of F and let $W_i^{(n)}$ be the X_i Winsorized outside of $(\xi_{\alpha_n}, \xi_{1-\beta_n}]$, i.e. $W_i^{(n)} = \max(\xi_{\alpha_n}, \min(X_i, \xi_{1-\beta_n}))$, $i = 1, \ldots, n$. In order to normalize T_n, we define two sequences

$$\mu_n = \mu(\alpha_n, 1 - \beta_n), \qquad \sigma_{W,n}^2 = \mathbf{Var}(W_i^{(n)}),$$

and assume that $\liminf_{n\to\infty} \sigma_{W,n} > 0$.

Let Φ denote the standard normal distribution function. Here is our main result on moderate deviations for intermediate trimmed means.

Theorem 2.1 ([10]) *Suppose that* $\mathbf{E}|X_1|^p < \infty$ *for some* $p > c^2 + 2$ $(c > 0)$. *In addition, assume that* $\frac{\log n}{\min(k_n, m_n)} \to 0$ *as* $n \to \infty$, *and that* $\max(\alpha_n, \beta_n) = O\big((\log n)^{-\gamma}\big)$, *for some* $\gamma > 2p/(p-2)$, *as* $n \to \infty$. *Then*

$$\mathbf{P}\Big(\frac{\sqrt{n}(T_n - \mu_n)}{\sigma_{W,n}} > x\Big) = [1 - \Phi(x)](1 + o(1)), \tag{1}$$

as $n \to \infty$, *uniformly in the range* $-A \leq x \leq c\sqrt{\log n}$ $(A > 0)$.

It is known that the intermediate trimmed mean T_n can serve as a consistent and robust estimator for $\mathbf{E}X_1$ (whenever it exists), and that the large and moderate deviations results for T_n can be helpful to construct more attractive confidence intervals for the expectation of X_1 than those that arise from the CLT.

Our next result concerns the asymptotic behavior of the first two moments of T_n and the possibility of replacing the normalizing sequences in (1) (in particular, replacing of μ_n by $\mathbf{E}X_1$).

Theorem 2.2 ([10]) *Suppose that the conditions of Theorem* 2.1 *are satisfied. Then*

$$n^{1/2}(\mathbf{E}T_n - \mu_n) = o\Big((\log n)^{-1}\Big), \quad \frac{\sigma_{W,n}}{\sigma} = 1 + o\big((\log n)^{-2}\big),$$
$$\frac{\sqrt{\mathbf{Var}(T_n)}}{\sigma_{W,n}/\sqrt{n}} = 1 + o\big((\log n)^{-1}\big), \quad \text{as } n \to \infty.$$

Moreover, μ_n *and* $\sigma_{W,n}$ *in relations* (1) *can be replaced respectively by* $\mathbf{E}T_n$ *and* σ *or* $\sqrt{n\mathbf{Var}(T_n)}$*, without affecting the result.*

Furthermore, if in addition

$$\max(\alpha_n, \beta_n) = o\big[(n \log n)^{-\frac{p}{2(p-1)}}\big],$$

then

$$n^{1/2}(\mathbf{E}X_1 - \mu_n) = o\Big((\log n)^{-1/2}\Big),$$

and μ_n *in* (1) *can be also replaced by* $\mathbf{E}X_1$*.*

3. Large and moderate deviations for trimmed L-statistics

In this section we consider the trimmed L-statistic given by

$$L_n = n^{-1} \sum_{i=k_n+1}^{n-m_n} c_{i,n} X_{i:n}, \quad \text{where} \quad c_{i,n} \in \mathbb{R}.$$

Let α_n, β_n denote the same sequences as before, and suppose now that

$$\alpha_n \to \alpha, \quad \beta_n \to \beta, \quad \text{as} \quad n \to \infty, \quad 0 < \alpha < 1 - \beta < 1,$$

i.e. we focus on the case of heavy trimmed L-statistic. Let J be a function defined in an open set I such that $[\alpha, 1-\beta] \subset I \subseteq (0,1)$. Define the trimmed L-statistic

$$L_n^0 = n^{-1} \sum_{i=k_n+1}^{n-m_n} c_{i,n}^0 X_{i:n} = \int_{\alpha_n}^{1-\beta_n} J(u) F_n^{-1}(u)\, du$$

with the weights $c_{i,n}^0 = n \int_{(i-1)/n}^{i/n} J(u)\, du$ generated by the function J.

To state our results, we need the following set of assumptions.

(i) *J is Lipschitz in I.*

(ii) F^{-1} *satisfies a Hölder condition of order* $0 < \varepsilon \le 1$ *in some neighborhoods* U_α *and* $U_{1-\beta}$ *of* α *and* $1-\beta$*.*

(iii) $\max(|\alpha_n - \alpha|, |\beta_n - \beta|) = O\big(n^{-\frac{1}{2+\varepsilon}}\big)$*, where* ε *is the Hölder index from condition* **(ii)**.

(iv) $\sum_{i=k_n+1}^{n-m_n} |c_{i,n} - c^0_{i,n}| = O(n^{\frac{1}{2+\varepsilon}})$, *where* ε *is as in conditions* **(ii)-(iii)**.

Define the distribution function of the normalized L_n:

$$F_{L_n}(x) = \mathbf{P}\{\sqrt{n}(L_n - \mu_n)/\sigma \le x\}, \tag{2}$$

where $\mu_n = \int_{\alpha_n}^{1-\beta_n} J(u)F^{-1}(u)\,du$, and the asymptotic variance

$$\sigma^2 = \int_{\alpha}^{1-\beta}\int_{\alpha}^{1-\beta} J(u)J(v)(\min(u,v) - uv)\,dF^{-1}(u)\,dF^{-1}(v).$$

Here is our main result on Cramér type large deviations for L_n.

Theorem 3.1 ([9]) *Suppose that* F^{-1} *satisfies condition* **(ii)** *for some* $0 < \varepsilon \le 1$ *and the sequences* α_n *and* β_n *satisfy* **(iii)**. *In addition, assume that the weights* $c_{i,n}$ *satisfy* **(iv)** *for some function* J *satisfying condition* **(i)**.

Then for every sequence $a_n \to 0$ *and each* $A > 0$

$$1 - F_{L_n}(x) = [1 - \Phi(x)](1 + o(1)), \tag{3}$$

as $n \to \infty$, *uniformly in the range* $-A \le x \le a_n n^{\varepsilon/(2(2+\varepsilon))}$.

Remark 3.1 Note that under somewhat stronger conditions **(iii')-(iv')** (cf. [9]) than **(iii)-(iv)**, the asymptotic variance σ in Theorem 3.1 can be replaced by $\sqrt{n\mathbf{Var}L_n}$, without affecting the result (see Theorem 1.2 [9]).

Corollary 3.1 *Suppose that the conditions of Theorem 2.1 are satisfied with* $\varepsilon = 1$, *i.e.* F^{-1} *is Lipschitz in some neighborhoods* U_α *and* $U_{1-\beta}$ *of* α *and* $1-\beta$. *Then for every sequence* $a_n \to 0$ *and each* $A > 0$ *relation* (3) *holds true, uniformly in the range* $-A \le x \le a_n n^{1/6}$.

Finally, we state our main results on probabilities of moderate deviations for L_n, i.e. the deviations in logarithmic ranges. We will need the following versions of conditions **(ii)-(iv)**.

(ii") *There exists a positive* ε *such that for each* $t \in \mathbb{R}$ when $n \to \infty$

$$F^{-1}\big(\alpha + t\sqrt{\log n/n}\big) - F^{-1}\big(\alpha\big) = O\big((\log n)^{-(1+\varepsilon)}\big),$$
$$F^{-1}\big(1 - \beta + t\sqrt{\log n/n}\big) - F^{-1}\big(1-\beta\big) = O\big((\log n)^{-(1+\varepsilon)}\big).$$

(iii") $\max(|\alpha_n - \alpha|, |\beta_n - \beta|) = O\big(\sqrt{\frac{\log n}{n}}\big)$, $n \to \infty$.

(iv") *For some* $\tilde{\varepsilon} > 0$ $\sum_{i=k_n+1}^{n-m_n} |c_{i,n} - c^0_{i,n}| = O\big(\frac{1}{\log^{\tilde{\varepsilon}} n}\sqrt{\frac{n}{\log n}}\big)$, $n \to \infty$. .

Theorem 3.2 ([8]) *Suppose that* F^{-1} *satisfies condition* **(ii")** *and that condition* **(iii")** *holds for the sequences* α_n *and* β_n. *In addition, assume that there exists a function* J *satisfying condition* **(i)** *such that* **(iv")** *holds for the weights* $c_{i,n}$. *Then relation* (3) *holds true, uniformly in the range* $-A \le x \le c\sqrt{\log n}$, *for each* $c > 0$ *and* $A > 0$.

Theorem 3.3 ([8]) *Suppose that the conditions of Theorem 3.2 hold true. In addition, assume that* $\boldsymbol{E}|X_1|^\gamma < \infty$ *for some* $\gamma > 0$. *Then*

$$\sqrt{n\mathbf{Var}(L_n)}/\sigma = 1 + O\big((\log n)^{-(1+2\nu)}\big),$$

where $\nu = \min(\varepsilon, \tilde{\varepsilon})$, ε, $\tilde{\varepsilon}$ *are as in conditions* **(ii")** *and* **(ii")** *respectively.*

Moreover, relation (3) *remains valid for each* $c > 0$ *and* $A > 0$, *uniformly in the range* $-A \le x \le c\sqrt{\log n}$, *if we replace* σ *in definition of* $F_{L_n}(x)$ (cf. (2)) *by* $\sqrt{n\mathbf{Var}(L_n)}$.

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