UDC 511.2:519.2

# Arithmetic Statistics, Probabilities and Langlands correspondence

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**Abstract.** We give a short review of some novel results in the area of arithmetic statistics. Then we present several results related to statistics of Artin-Schreier covering over fields of finite characteristic and over the ring of rational integers, which support our computations and some of which follow from general theorems by N. Katz and others about global monodromy of Kloosterman sheaves. In this framework, we can reprove known results for two parameter Kloosterman sums. In a more general framework, connections among arithmetic statistics, probabilities and Langlands correspondence are investigated.

**Keywords:** analytical methods in probability theory, Kloosterman sum, Artin-Schreier covering, $L-$function, Sato-Tate density.

## 1. Introduction

Classical works by P.L. Chebyshev, A.O. Gel'fond, B.V. Gnedenko, A.N. Kolmogorov, Yu.V. Linnik, A.G. Postnikov, and other researchers, demonstrate the fruitfulness of the interaction of analytical and computational methods in solving problems in probability theory and its applications. In the framework of this approach A.V. Malyshev have developed Linnik's ergodic method in number theory. The fruitfulness of this approach to appropriate problems of probability theory and reliability theory have demonstrated by works of A.D. Solovyev and his colleagues.

Nowadays some aspects of the investigations are reflecting in the areas of arithmetic statistics and Cohen-Lenstra heuristics. These two areas are mathematical approaches which were founded in order to throw light on the limiting behavior of (number-theoretic and group) objects in families [1–4].

Arithmetic statistics of subschemes of abelian varieties over finite fields have investigated in paper [2] .

Arithmetic statistics and the probability that a complete intersection is smooth have investigated in the paper by A. Bucur, K. Kedlaya in [3]. Let $X$ be a smooth subscheme of a projective space over a finite field. In the case of a single hypersurface of large degree, the probability that its intersection with $X$ is smooth of the correct dimension was computed by B. Poonen in [4]. A. Bucur, K. Kedlaya generalize Poonen's theorem to the case of complete intersections. They then relate this result to the probabilistic model of rational points of such complete intersection. Briefly, the general idea of the proof of authors Theorem is a sieve over the set of closed points of $X$ similar to Poonen's proof of Theorem 1.1. in which authors separately analyze the contribution of points of low degree, medium degree and high degree. Authors also give an interesting corollary that the number of rational points on a random smooth intersection of two surfaces in projective 3-space is strictly less than the number of points on the projective line. Finally, they indicate some other directions in which one can probably generalize Poonen's results from hypersurfaces to complete intersections and make the conjecture, generalizing the conditional Poonen's Theorem 5.5. There are another interesting papers on arithmetic statistics.

In our communication we present several results related to statistics of Artin-Schreier covering over fields of finite characteristic and over *Spec* **Z**, which support our computations and which follow from general theorems by N. Katz and others about global monodromy of Kloosterman sheaves. In this framework, we can reprove known results for two parameter Kloosterman sums. In a more general framework, connections among arithmetic statistics, probabilities and Langlands correspondence are investigated.

## 2. Equidistribution and $L-$functions

Let $X$ be a compact topological space and $C(X)$ be the Banach space of continuous complex-valued functions on $X$. For $f \in X$ let its norm $||f|| = sup_{x\in X}|f(x)|$. Let $\delta_x$ be the Dirac measure associated to $x \in X$ : $\delta_x(f) = f(x)$. For a sequence $(x_n), n \geq 1$ let $\mu_n = \frac{\delta_{x_1}+\cdots+\delta_{x_n}}{n}$. Let $\mu$ be a Radon measure on $X$.

The sequence $(x_n)$ is said to be equidistributed with respect to measure $\mu$ (or$\mu$-equidistributed) if $\mu_n \to \mu$ (weakly) as $n \to \infty$.

Let $G$ be a compact group and let $X$ be the space of conjugacy classes of G Let $x_v$, $v \in \Sigma$ be a family of elements of $X$ indexed by a denumerable set $\Sigma$.

**Theorem** (Chebotarev, Artin, Serre). The elements $x_v$, $v \in \Sigma$ are equidistributed for the normalized Haar measure of $G$ if and only if the $L-$functions relative to the non trivial irreducible characters of $G$ are holomorphic and non zero at $s = 1$.

## 3. Experimental distribution of Kloosterman sums and Sato-Tate density

**Definition 1** *Let*

$$T_p(c,d) = \sum_{x=1}^{p-1} e^{2\pi i(\frac{cx+\frac{d}{x}}{p})},$$

$$1 \leq c,d \leq p-1;\ x,c,d \in \mathbf{F}_p^*$$

*be a Kloosterman sum.*

By A. Weil estimate $T_p(c,d) = 2\sqrt{p}\cos\theta_p(c,d)$.

In seventies the author of the communication have computed the distribution of angles $\theta_p(c,d)$ (mainly for the case $c = d = 1$, prime $p$ runs from 2 in interval $[2, 10000]$, and unsystematically for some prime from the interval with constant $p$ and varying $1 \leq c, d \leq p-1$).

There are possible two distributions of angles $\theta_p(c,d)$ on semiinterval $[0,\pi)$ :

a) *$p$ is fixed and $c$ and $d$ varies over $\mathbf{F}_p^*$; what is the distribution of angles $\theta_p(c,d)$ as $p \to \infty$ ;*

b) *$c$ and $d$ are fixed and $p$ varies over all primes not dividing $c$ and $d$.*

*Conjecture 1.* In the case b) when $p$ varies over all primes then angles $\theta_p(1,1)$ are distributed on the interval $[0,\pi)$ with the Sato-Tate density $\frac{2}{\pi}\sin^2 t$.

## 4. Artin-Schreier covering and Kloosterman sums

Let $k$ be a field of characteristic $p > 0$ and let $X$ be a nonsingular projective variety defined over $k$. In most cases $k = \mathbb{F}_q, q = p^s$ for some positive integer $s$.

An Artin-Schreier covering of $X$ is a finite morphism $\pi : Y \to X$ from a normal variety $Y$ onto $X$ such that the field extension $k(Y)/k(X)$ is an Artin-Schreier extension. This extension is defined by the Artin-Schreier equation $y^p - y = f$ and $f \in k(X)$.

*Example 1.* The equation in affine form $y^p - y = cx + \frac{d}{x},\ c, d \in \mathbb{F}_p^*$ defines an Artin-Schreier covering of the projective line.

*Example 2.* The equation in affine form $y^p - y = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,\ a_i \in \mathbb{F}_p$ defines an Artin-Schreier curve and the respective covering of the projective line.

**Proposition 1** *The Kloosterman sum $T_p(c,d)$ is defined by the Artin-Schreier covering*

$$y^p - y = cx + \frac{d}{x},\ c, d \in \mathbb{F}_p^*.$$

**Proposition 2** *See [5, 11]. Let $X$ be reduced absolutely irreducible nonsingular projective algebraic curve of genus $g$ defined over a finite field $\mathbb{F}_q$. Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. We may regard $X$ as an algebraic curve over $\overline{\mathbb{F}}_q$. In this case on can attach to $X$ the Jacobean $J(X)$ of $X$. Formal completion of the variety defines the commutative formal group $F$ of dimension $g$.*

**Proposition 3** *Under the conditions of the Proposition 3 there is an algebraic construction of the continuation of the Artin-Schreier covering $\pi : Y \to X$ to the map $\pi^{\text{‘}} : J(Y) \to J(X)$.*

Let $l$ be a prime such that $l \neq p$. Let $T_l(A)$ be the Tate module of the abelian variety $A$. In the case of the Jacobean $J(X)$ the Tate module is a free module of the rank $2g$ over $l-$adic numbers.

## 5. Sato-Tate type conjectures and elements of Langlands program

Geometric analogous of Sato-Tate conjecture (Sato-Tate conjecture over functional fields) are investigated and proved by B. Birch [7] and by H. Yoshida [8].

Let (in P. Deligne notations) $X$ be a scheme of finite type over $\mathbf{Z}$, $|X|$ the set of its closed points, and for each $x \in |X|$ let $N(x)$ be the number of points of the residue field $k(x)$ of $X$ at $x$. The Hasse-Weil zeta-function of $X$ is, by definition $\zeta_X(s) = \prod_{x \in |X|}(1 - N(x)^{-s})^{-1}$.

The Hasse-Weil zeta function of $E$ over $\mathbf{Q}$ (an extension of numerators of $\zeta_E(s)$ by points of bad reduction of $E$) is defined over all primes $p$: $L(E(\mathbf{Q}), s) = \prod_p (1 - a_p p^s + \epsilon(p) p^{1-2s})^{-1}$, here $\epsilon(p) = 1$ if $E$ has good reduction at $p$, and $\epsilon(p) = 0$ otherwise.

A cusp (parabolic) form of weight $k \geq 1$ and level $N \geq 1$ is a holomorphic function $f$ on the upper half complex plane $\mathbf{H}$ .

Langlands conjectured that some symmetric power L-functions extend to an entire function and coincide with certain automorphic $L-$functions. Sato-Tate conjecture, now Clozel–Harris–Shepherd-Barron–Taylor Theorem:

**Theorem** (Clozel, Harris, Shepherd-Barron, Taylor). Suppose $E$ is an elliptic curve over $\mathbf{Q}$ with non-integral $j$ invariant. Then for all $n > 0; L(s; E; Sym^n)$ extends to a meromorphic function which is holomorphic and non-vanishing for $Re(s) \geq 1 + n/2$.

These conditions and statements are sufficient to prove the Sato-Tate conjecture.

Under the prove of the Sato-Tate conjecture the Taniyama-Shimura-Weil conjecture oriented methods of A. Wiles and R. Taylor are used.

Recall the main (and more stronger than in Wiles and in Wiles-Taylor papers) result by C. Breuil, B. Conrad, F. Diamond, R. Taylor.

**Taniyama-Shimura-Weil conjecture - Wiles Theorem**. For every elliptic curve $E$ over $\mathbf{Q}$ there exists $f$, a cusp form of weight 2 for a subgroup $\Gamma_0(N)$, such that $L(f, s) = L(E(Q), s)$.

## 6. Distribution of Kloosterman sums in functional case

**Theorem 1** *For any sequence of prime finite fields* $\mathbf{F}_p$*, when* $p$ *tends to* $\infty$*,* $1 \leq c, d \leq p-1$*;* $x, c, d \in \mathbf{F}_p^*$*. the distribution of angles* $\theta_p(c, d)$ *on semiinterval* $[0, \pi)$ *tends to the equidistribution with the density* $\frac{2}{\pi} \sin^2 t$*.*

### 6.1. Sketch of the proof

( follow to [9]. Very roughly.)

a) Construction of $l-$adic representations, $l-$adic cohomologies and corresponding sheaves.

b) Introduction and investigation of corresponding $L-$-functions and their zeroes.

c) Testing that the given representation is pure. Computation of the weight of the representation. Selection of representations.

d) Applications of Deligne's results.

## 7. Conclusions

The short review of some novel results in the area of arithmetic statistics is given. Several results related to statistics of Artin-Schreier covering over fields of finite characteristic and over *Spec* $\mathbf{Z}$, which support our computations and some of which follow from general theorems by N. Katz and others about global monodromy of Kloosterman sheaves are presented. The sketch of the proof of the analog of N. Katz theorem for two parameter Kloosterman sums is given.

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