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About the thinning of a flow with limited aftereffect and different interarrivals distributions

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Abstract. In the talk, we present some analytical results obtained for the probability characteristics of a thinning of a flow with limited aftereffect. The thinning is performed according to a given function which depends on evolution time and on the number of customers in the thinned flow and the number of lost customers in the original flow. The characteristics are obtained in the form of Laplace-Stieltjes transforms defined by a system of recurrence equations using inversion of Laplace-Stieltjes transforms. A special case is also considered.

Keywords: flow with limited aftereffect, thinning of a flow, time-depend function of thinning, Laplace-Stieltjes transform, inversion of Laplace-Stieltjes transform .

1. Introduction

Let us recall some results about thinning of flows. Renyi [1] in 1956 proved the first theorem on thinning of a renewal flow. A customer joins the thinned flow with a constant probability q, and with probability $p = 1-q$ he doesn't. By changing the time scale the flow can be made constant intensity. Assume that the thinning is performed n times with different probabilities $q_1, \ldots, q_n$. Then, provided that $n \to \infty$ and $q_1, \ldots, q_n \to \infty$, the thinned flow converges to a Poisson flow. Belyaev Yu.K. in an addedum «Random flows and renewal theory» to the Russina translation of the book by D. Cox and V. Smith [2] wrote about the preservation of Poissonian property by the flow under the thinning of the original Poisson flow.

Belyaev Yu. K. [3] generalized this fact to arbitrary flows. Gnedenko B.V. and Kovalenko I.N. in [4] generalized Belyaev's theorem to the case of a non-stationary limiting flow. A. D. Soloviev in [5] in 1971 demonstrated that asymptotically the time of the first occurrence of a rare event in a regenerative process with appropriate normalization tends to an exponential random variable with parameter 1. Some other results about thinned flows can be found in [8–12].

For all of these works the aim was to obtain the limit theorems in the case of infinite thinning and under appropriate normalization. The common point for the above works was the fact that the thinning was carried out according to procedures that were time-independent. The novelty of this work is that the thinning is performed according to well-defined time-dependent procedures.

2. The problem statement

In his article [6] V. Smith studied a flow of customers with different probability distributions of interarrival intervals. A.Ya. Khinchin [7] named such flows of customers the flows with limited aftereffect. The present article considers a flow of customers with limited aftereffect and thinning. The first customer in this flow arrives at a random time with the probability distribution function $F_1(x)$.

The time interval between the arrival of the first customer and the arrival of the second customer has a probability distribution function $F_2(x)$, and in general, the time interval between the arrival of the $(i-1)$-st customer and the arrival of the i-th customer has a probability distribution function $F_i(x)$, $i = 2, 3, \ldots$.

The thinning meant the following. If the customer arrives at time t, the number of customers joined to the thinned flow by this time equals $i-1$, and the number of customers from the source flow who didn't join (wad lost) equals j, then the customer either joins the thinned flow with probability $P_{i,j}(t)$, or is lost with the complementary probability. The next interarrival time has distribution function $F_{i+j}(x)$. The functions $P_{i,j}(t)$ are assumed to be known. We aim to find the probability distribution of the number of customers who joined the thinned stream up to an arbitrary time t, assuming that at times $t = 0$ the number of customers in thinned flow was zero.

3. Solution

We introduce the following notation: $\nu(t)$ – the number of customers who joined the thinned flow, $\nu_0(t)$ – the number of lost customers from the source flow with limited aftereffect, $\xi(t)$ – the elapsed time between t and the next arrival from the source flow with limited aftereffect.

First, consider the process $\zeta(t) = (\nu(t), \xi(t))$. This process will not be Markovian random process, since its values after time t will not depend only on values $\nu(t)$ and $\xi(t)$, but will also depend on the number of lost customers up to the time t from the source flow with limited aftereffect. This is because the lost customers shift time instants of customer arrivals to the thinned stream on the time axis.

Indeed, consider two consecutive times ξ_0 and $\xi_0 + \xi$. Let us assume that both times the customers arrived from the source flow with limited aftereffect have not joined the thinned flow, the probability of this event equals $(1 - P_{0,0}(\xi_0))(1 - P_{0,0}(\xi_0 + \xi))$. If we consider the process $\zeta(t) = (\nu(t), \xi(t))$, the probability of this event is equal to $(1 - P_0(\xi_0))^2$, as the shift on the time axis by the amount ξ will not be considered because the value $\xi(t)$ is not taken into account.

Let us now consider the process $\zeta(t) = (\nu(t), \nu_0(t), \xi(t))$. This process already takes into account the fact that the lost customers shift points in time of customers arrivals to the thinned flow on the time axis. Therefore, the process $\zeta(t) = (\nu(t), \nu_0(t), \xi(t))$ will already be a Markov random process, its values after the time t will depend on $\nu(t)$, $\nu_0(t)$ and $\xi(t)$, i.e. will not depend on its states before time t. We introduce the notation $\varphi_{i,j}(t,x) = P(\nu(t) = i, \nu_0(t) = j, \xi(t) < x)$, $\varphi_{i,j}(t) = \varphi(i,j)(t,\infty)$, $i = 0, 1, \ldots$, $j = 0, 1, \ldots$.

The problem is to find the probability distribution $\varphi_{i,j}(t)$ for the number of accepted customers to the thinned flow by a fixed time t.

First, we wiill find the joint probability distribution for the number of accepted customers in the thinned flow and the elapsed time till next arrival, i.e. $\varphi_{i,j}(t,x)$. Initially, number customers is zero (at time $t = 0$).

We derive a system of differential equations for the desired quantities $\varphi_{i,j}(t,x)$. We have the following system of difference equations

$$\begin{aligned}
&\varphi_{0,0}(t+\Delta t, x-\Delta t) = \varphi_{0,0}(t,x) - \varphi_{0,0}(t,\Delta t),\\
&\varphi_{0,j}(t+\Delta t, x-\Delta t) = \varphi_{0,j}(t,x) - \varphi_{0,j}(t,\Delta t)\\
&\qquad + \varphi_{0,j-1}(t,\Delta t)(1 - P_{0,j-1}(t))F_j(x),\ j>0,\\
&\varphi_{i,0}(t+\Delta t, x-\Delta t) = \varphi_{i,0}(t,x) - \varphi_{i,0}(t,\Delta t)\\
&\qquad + \varphi_{i-1,0}(t,\Delta t)P_{i-1,0}(t))F_i(x),\ i>0,\\
&\varphi_{i,1}(t+\Delta t, x-\Delta t) = \varphi_{i,1}(t,x) - \varphi_{i,1}(t,\Delta t) + \varphi_{i-1,1}(t,\Delta t)\\
&\qquad \times P_{i-1,1}(t)F_{i1}(x) + \varphi_{i,0}(t,\Delta t)(1 - P_{i,0}(t))F_{i+1}(x),\ i>0,\\
&\varphi_{i,j}(t+\Delta t, x-\Delta t) = \varphi_{i,j}(t,x) - \varphi_{i,j}(t,\Delta t) + \varphi_{i-1,j}(t,\Delta t)\\
&\quad \times P_{i-1,j}(t)F_{i+j}(x) + \varphi_{i,j-1}(t,\Delta t)(1 - P_{i,j-1}(t))F_{i+j}(x),\ i>0, j>1.
\end{aligned}$$

This yields the following system of differential equations for $\varphi_{i,j}(t,x)$:

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\varphi_{0,0}(t,x) - \tfrac{\partial}{\partial x}\varphi_{0,0}(t,x) = -\tfrac{\partial}{\partial x}\varphi_{0,0}(t,0),\\
&\tfrac{\partial}{\partial t}\varphi_{0,j}(t,x) - \tfrac{\partial}{\partial x}\varphi_{0,j}(t,x) = -\tfrac{\partial}{\partial x}\varphi_{0,j}(t,0)\\
&\qquad + \tfrac{\partial}{\partial x}\varphi_{0,j-1}(t,0)(1 - P_{0,j-1}(t))F_j(x),\ j>0,\\
&\tfrac{\partial}{\partial t}\varphi_{i,0}(t,x) - \tfrac{\partial}{\partial x}\varphi_{i,0}(t,x) = -\tfrac{\partial}{\partial x}\varphi_{i,0}(t,0)\\
&\qquad + \tfrac{\partial}{\partial x}\varphi_{i-1,0}(t,0)P_{i-1,0}(t))F_i(x),\ i>0,\\
&\tfrac{\partial}{\partial t}\varphi_{i,1}(t,x) - \tfrac{\partial}{\partial x}\varphi_{i,1}(t,x) = -\tfrac{\partial}{\partial x}\varphi_{i,1}(t,0)\\
&\qquad + \tfrac{\partial}{\partial x}\varphi_{i-1,1}(t,0)P_{i-1,1}(t))F_{i+1}(x)\\
&\qquad + \tfrac{\partial}{\partial x}\varphi_{i,0}(t,0)(1 - P_{i,0}(t))F_{i+1}(x),\ i>0,\\
&\tfrac{\partial}{\partial t}\varphi_{i,j}(t,x) - \tfrac{\partial}{\partial x}\varphi_{i,j}(t,x) = -\tfrac{\partial}{\partial x}\varphi_{i,j}(t,0)\\
&\qquad + \tfrac{\partial}{\partial x}\varphi_{i-1,j}(t,0)P_{i-1,j}(t))F_{i+j}(x)\\
&\qquad + \tfrac{\partial}{\partial x}\varphi_{i,j-1}(t,0)(1 - P_{i,j-1}(t))F_{i+j}(x),\ i>0, j>1.
\end{aligned}$$

We introduce the notations:

$$\tilde{\varphi}_0^{(0)}(s)=\int\limits_0^\infty e^{-sx}\,dF_1(x)=\tilde{\varphi}_1(s),\quad \tilde{\varphi}_i(s)=\int\limits_0^\infty e^{-sx}\,dF_i(x)=\tilde{\varphi}_1(s),\ i>0,$$

$$\tilde{\varphi}_{i,j}(u,s)=\int\limits_0^\infty\int\limits_0^\infty e^{-sx-ut}\,d_x\varphi_{i,j}(t,x)\,dx,$$

$$\tilde{\varphi}_{i,j}(u)=\int\limits_0^\infty e^{-ut}\varphi_{i,j}(t)\,dt,\ \tfrac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)=\int\limits_0^\infty e^{-ut}\tfrac{\partial}{\partial x}\varphi_{i,j}(t,0)\,dt,$$

$$\tfrac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j}(u,0)=\int\limits_0^\infty e^{-ut}\tfrac{\partial}{\partial x}\varphi_{i,j}(t,0)P_{i,j}(t)\,dt,\ i=0,1,\ldots,\ j=0,1,\ldots.$$

The following theorem holds.

Theorem 1. *The Laplace-Stieltjes transforms $\tilde{\varphi}_{i,j}(u,s)$ of functions $\varphi_{i,j}(t,x)$ satisfy the following relations:*

$$\begin{aligned}\tilde{\varphi}_{0,0}(u,s)&=(u-s)^{-1}(\tilde{\varphi}_1(s)-\tilde{\varphi}_1(u)),\\ \tilde{\varphi}_{i,j}(u,s)&=(u-s)^{-1}\big(-\tfrac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)+\tfrac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\tilde{\varphi}_{i+1}(s)\\ &\qquad+\tfrac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\tilde{\varphi}_{i+j}(s)-\tfrac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\tilde{\varphi}_{i+j}(s)\big),\ i>0,j>1\end{aligned}$$

where $\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)$ are determined sequentially from the following recurrent equations

$$\begin{aligned}\tfrac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)&=\tilde{\varphi}_{i+1}(u)\big(\tfrac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)+\tfrac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)-\tfrac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\big)\\ &=\tilde{\varphi}_{i+1}(u)\Bigg(\int\limits_0^\infty e^{-ut}\tfrac{\partial}{\partial x}\varphi_{i-1,j}(t,0)P_{i-1,j}(t)\,dt\\ &\qquad+\int\limits_0^\infty e^{-ut}\tfrac{\partial}{\partial x}\varphi_{i,j-1}(t,0)(1-P_{i,j-1}(t))\,dt\Bigg).\end{aligned}$$

References

1. *Renyi A.* A Characterization of Poisson Processes // Magyar Tud. Akad. Mat. Kutató Int. Közl. — 1956. — V. 1, no. 4. — P. 519–527.
2. *Cox, D.R., Smith W.L.* Renewal Theory. — Soviet Radio, 1967.

3. *Belyaev Yu.K.* Limit theorems for Rearing Flows // Theory of Probability and its Applications. — 1963. — V. 8, no. 2. — P. 175–184.
4. *Gnedenko B.V., Kovalenko, I.N.* Introduction to Queueing Theory, Second Edition. — Birkhauser, 1989.
5. *Soloviev A.D.* Asymptotic Behavior of the First occurrence of a Rare Eevent in a Regenerating Process // Engineering Cybernetics. — 1971. — No. 6. — P. 79–89.
6. *Smith W.L.* On some general renewal theorems for non-identically distributed Variables // Proceedings of the 4-th Berkeley Symposium. — 1961. — P. 467–514.
7. *Khintchine A.Y.* Mathematical Methods in the Theory of Queueing. — Charles Griffin and Co., 1960.
8. *Streit R.l.* Poisson Point Processes: Imaging, Tracking, and Sensing. — Springer, 2010.
9. *Serfozo R.* A Course in Applied Stochastic Processes. — Springer, 2009.
10. *Assuncao R.M., Ferrari P.A.* Independence of Thinned Processes Characteristics the Poisson Process: an Elementary Proof and a Statistical Application // Test. — 2007. — V. 16, no. 2. —- P. 333–345.
11. *Kushnir A.O.* Asymptotic Behavior of a Renewal Process Trinned by an Alternating Process // Cybernetics and Systems Analysis. — 1993 — V. 29, no. 1. — P. 20–25.
12. *Gurel-Gurevich O., Peled R.* Poisson Thickening // Israel Journal of Mathematics —2013. — V. 196, no. 1. — P. 215–234.