UDC 519.21

Steady state for the critical branching random walk with the general number of offsprings

E. Chernousova*, S. Molchanov†‡

* *Department of Mathematical Basics of Control, Moscow Institute of Physics and Technology (State University), 9 Institutskiy per., Dolgoprudny, Moscow Region, 141701, Russia*

† *Laboratory of Stochastic Analysis and its Applications, Higher School of Economics, Shabolovka str. 28/1, Moscow , Russia*

‡ *Department of Mathematics and Statistics, University of North Carolina, Charlotte, NC 28223, USA*

Abstract. We prove the existence of steady state for branching random walk with arbitrary number of offsprings.

Keywords: branching random walk, steady state, Carleman's conditions.

1. Introduction

This paper is natural continuation of the publication [1], where was proposed the new approach to the problem of the steady state for the critical contact process. The very fact of the existence of such state was not new (starting from the works of the German group [2], [3], based on the ideas by R. Dobrushin, see also papers [4] by Yu. Kondratiev with collaborators). Usually the proof of the convergence to the steady state was based on the forward Kolmogorov equation for the correlation functions, but in [1] the analysis was based on the backward equations.

To simplify the calculations all papers mentioned above included assumption on binary birth process: each particle either dies at the time interval $(t, t + dt)$ with probability $\mu\, dt + o(dt)$ or splits into two particles (or, one can say, the parental particle produces one offspring) with probability $\beta\, dt + o(dt)$. The criticality corresponds to the equality $\mu = \beta$.

The offsprings either can start its evolution from the position of the parental particle or can make the random jump with some distribution. The situation when there is no migration of the particles but only the random jumps of offsprings (seeds) corresponds to the 'forest' model. Note that in this situation the assumption on the binary splitting looks especially artificially.

We'll consider the general model of the branching random walk on $\mathbb{Z}^d$, $d \geq 1$, where the evolution of the particles include the migration, death with some mortality rate μ, the splitting with arbitrary member of offsprings (seeds) and their distribution around parental particle with some law.

2. Description of the model. Main results

We use $N(t,y)$ to designate particle field, that is, global population, $t \geq 0$, $y \in \mathbb{Z}^d$. $n(t,y;x)$ designate a subpopulation generated by a single initial particle at $x \in \mathbb{Z}^d$ at $t=0$. The subpopulations are independent and

$$N(t,y) = \sum_{x \in \mathbb{Z}^d} n(t,y;x), N(0,y) \equiv 1.$$

The evolution of each subpopulation includes random walk of each particle with generator:

$$\kappa \mathcal{L}_a \psi = \kappa \sum_{v \neq 0} a(v)[\psi(x+v) - \psi(x)],$$

where $a(v) = a(-v)$ (symmetry) and $\sum_{v \in \mathbb{Z}^d, v \neq 0} a(v) = 1$ (normalization). It also includes the reaction of annihilation or death with rate μ and splitting of the particle into l particles with rates β_l where $l \geq 2$. In such splitting, one offspring (it can be considered as parent particle) remains at the same point and others $l-1$ particles jump independently from x to $x+v$ with distribution $b(v)$ where $b(v) = b(-v)$ and $\sum_{v \in \mathbb{Z}^d} b(v) = 1$.

Let's introduce the generation function of individual subpopulation $u_z(t,x;y) = \mathbb{E} z^{n(t,y;x)}$. Note the reversal of the start x and destination y in this notation, because we look at this function as the function of t and x variable. For every fixed $y \in \mathbb{Z}^d$ $u_z(t,x;y)$ satisfies the backward Kolmogorov equations:

$$\frac{\partial u_z}{\partial t} = \kappa \mathcal{L}_a u_z - \left(\mu + \sum_{l=2}^{\infty} \beta_l \right) u_z + \mu + u_z \sum_{l=2}^{\infty} \beta_l \left(u_z * b \right)^{l-1} \qquad (1)$$

with initial condition: $u_z(0,x;y) = z$ if $x = y$ and $u_z(0,x;y) = 1$ else. Here we designate

$$u_z * b = \sum_{v \in \mathbb{Z}^d} u_z(t, x+v; y) b(v).$$

From this we can lead the equation for factorial moments

$$m_k(t,x;y) = \mathbb{E}\left[n(n-1)\cdots(n-k+1) \right] = \left. \frac{\partial^k u_z}{\partial z^k} \right|_{z=1} (t,x;y),$$

where $n = n(t,y;x)$, $k = 1, 2, \ldots$.

For the first moment we have:

$$\frac{\partial m_1}{\partial t} = \left(\kappa\mathcal{L}_a + \sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b\right) m_1 + \left(\sum_{l=2}^{\infty}(l-1)\beta_l - \mu\right) m_1$$
$$m_1(0,x;y) = \delta(x-y).$$

In the case $\mu = \sum_{l=2}^{\infty}(l-1)\beta_l$ (which is critical) this equation has a simple form:

$$\frac{\partial m_1}{\partial t} = \left(\kappa\mathcal{L}_a + \sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b\right) m_1$$
$$m_1(0,x;y) = \delta(x-y)$$

with fundamental solution $m_1(t,x;y) = p(t,x,y)$, where $p(t,x,y)$ is a conditional probability the particle started from $x \in \mathbb{Z}^d$ during time $t > 0$ go to $y \in \mathbb{Z}^d$ when random walk defines symmetric isotropic generator $\kappa\mathcal{L}_a + \sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b$.

As is very well known a critical Galton-Watson process,

$$\nu_x(t) = \sum_{y\in\mathbb{Z}^d} n(t,y;x)$$

for each $x \in \mathbb{R}^d$, degenerates almost sure with rate $1 - \frac{2}{t\sum_{l=2}^{\infty} l(l-1)\beta_l}$ and with the expected number of particles at $t > 0$:

$$\mathbb{E}\nu_x(t) = \sum_{y\in\mathbb{Z}^d} \mathbb{E}n(t,y;x) = \sum_{y\in\mathbb{Z}^d} p(t,x,y) = 1.$$

More accurate (see [5]) the limit law for nondegenerated population has a form:

$$\lim_{t\to\infty} \mathbb{P}\left\{\frac{2\nu_x(t)}{t\sum_{l=2}^{\infty} l(l-1)\beta_l} > a \mid \nu_x(t) > 0\right\} = e^{-a}.$$

That is if a population does not vanish then a total number of particles is large.

Our goal is to prove the existence of a stationary limiting distribution in the critical case. For these we will show that the moments meet Carleman conditions, thus, that the moments are sufficient to establish a unique limiting distribution.

In the case $\mu = \sum_{l=2}^{\infty}(l-1)\beta_l$ the equation for generation function (1) has a form:

$$\frac{\partial u_z}{\partial t} = \kappa\mathcal{L}_a u_z + \sum_{l=2}^{\infty}(l-1)\beta_l - \left(\sum_{l=2}^{\infty} l\beta_l\right) u_z + u_z \sum_{l=2}^{\infty} \beta_l \left(u_z * b\right)^{l-1}$$

For all $k \geq 2$ we can lead the equations for the k-th factorial moments:

$$\frac{\partial m_k}{\partial t} = \left(\kappa \mathcal{L}_a + \sum_{l=2}^{\infty} (l-1)\beta_l \mathcal{L}_b \right) m_k +$$

$$\sum_{l=2}^{\infty} \beta_l \sum_{n=1}^{k-1} \frac{m_n}{n!} \sum_{\substack{\sum_{s=1}^{l-1} j_s = k-n, \\ j_s \geq 0}} \frac{k!}{j_1! \cdots j_l!} (m_{j_1} * b) \cdot \ldots \cdot \left(m_{j_{l-1}} * b \right) +$$

$$\sum_{l=2}^{\infty} \beta_l \sum_{\substack{\sum_{s=1}^{l-1} j_s = k, \\ 0 \leq j_s \leq k-1}} \frac{k!}{j_1! \cdots j_l!} (m_{j_1} * b) \cdot \ldots \cdot \left(m_{j_{l-1}} * b \right)$$

where it is assumed that $m_0(t,x;y) \equiv 1$ and the initial condition $m_k(0,x;y) = 0$ for $k \geq 2$.

Next without detraction from generality it is assumed that $y = 0$. We will investigate the limiting distribution of the process $N(t,0)$.

Lemma 1. If for all $l \geq 2$ $\beta_l \leq \beta\delta^l$ for some $\beta > 0$, $\delta \in (0,1)$, then $m_k(t,x;0) \leq k! B^{k-1} D_k p(t,x,0)$ for all $k \geq 1$ where

$$B = \max \left\{ 1, \beta \int_0^{\infty} p(s,0,0) ds \right\}$$

and the sequence D_k is recurrently defined as: $D_1 = 1$, for $k \geq 2$

$$D_k = \sum_{l=2}^{\infty} \delta^l \sum_{n=1}^{k-1} D_n \sum_{i=1}^{l-1} \binom{l-1}{i} \sum_{\substack{\sum_{s=1}^{i} j_s = k-n, \\ j_s \geq 1}} D_{j_1} \cdot \ldots \cdot D_{j_i} +$$

$$\sum_{l=2}^{\infty} \delta^l \sum_{i=2}^{l-1} \binom{l-1}{i} \sum_{\substack{\sum_{s=1}^{i} j_s = k, \\ j_s \geq 1}} D_{j_1} \cdot \ldots \cdot D_{j_i}.$$

Lemma 2. The sequence D_k increases not faster than geometrically.

Corollary. Let's for all $l \geq 2$ it holds the upper bound $\beta_l \leq \beta\delta^l$ for some $\beta > 0$, $\delta \in (0,1)$. Then $m_k(t,x;0) \leq c^k k! p(t,x,0)$ for all $k \geq 1$

To make sure that Carleman's condition is hold we note that due to independence of subpopulations the cumulants κ_k of all population $N(t,y)$ is the sum of cumulants of $n(t,x,y)$ (remind that $k-$th cumulant of the discrete random value X defines as

$$\kappa_k(X) = \left. \frac{d^k \ln \mathbb{E} z^X}{dz^k} \right|_{z=1},$$

i.e. Taylor expansion of the log-generation function in the neighborhood of $z = 1$ can be expressed as

$$\ln \mathbb{E} z^X = \sum_{k=0}^{\infty} \frac{\kappa_k(X)}{k!} (z-1)^k).$$

Moreover we can evaluate the cumulants from factorial moments and moments from cumulants.

The main result is

$$N(t, \cdot) \stackrel{Law}{\to} N(\infty, \cdot), \quad t \to \infty,$$

i.e. the probability distribution of the population converges to a limiting or stationary distribution.

3. Conclusions

It is considered branching random walk in $\mathbb{Z}^d$, $d \geq 1$, satisfying some conditions which mean zero drift of random migration process of particles and symmetric spreading of offsprings about parental particle. We state that if the initial population consists in the single particle at each point of $\mathbb{Z}^d$, the rate of mortality coincides with average numbers of new particles per time (critical case) and the tail of distribution of numbers of offsprings decreases at least geometrically then the probability distribution of the population converges to a limiting or stationary distribution.

Acknowledgments

The study has been funded by the Russian Science Foundation (project No. 17-11-01098)

References

1. *Molchanov S., Whitmeyer J.* Stationary distributions in KPP-type models with an infinite number of particles // Math. Popul. Stud. (Forthcoming)
2. *Debes H., Kerstan J., Liemant A., Matthes K.* Verallgemeinerungen eines Satzes von Dobruschin. I // Math. Nachr. — 1970. — Vol. 47. — P. 183–244.
3. *Liemant A.* lnvariante zufallige Punktfolgen // Wiss. Z. Friedrich-Schiller-Univ. Jena 19. — 1969. — P. 361–372.
4. *Kondratiev Y., Kutovyi O., Pirogov S.* Correlation functions and invariant measures in continuous contact model // Infn. Dimens. Anal. Quantum Probab. Relat. Top. — 2008. — Vol. 11. — P. 231–258.
5. *Karlin S.* A First course in stochastic prosesses. — Academic Press, New York, London, 1968.