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Priority systems with orientation. Analytical and numerical results

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Abstract. A class of priority queueing systems with non-zero switchover times is considered. Some performance characteristics such as distributions of busy periods, conditions of stationarity, traffic coefficients, etc. are presented. Numerical algorithms for their modelling are developed.

Keywords: priority queueing systems, semi-Markov orientation, busy period, traffic coefficient, numerical algorithm.

1. Introduction

Mathematical models of queueing theory play an important role in the solution of a wide range of topical applied problems, arising from the rational organization of industries, polyclinics and hospitals, resource management, information flows, transport, information sorting and processing, and other branches of human activity. Especially bright and impressive is the application of this theory in the analysis and design of various modern networks and their components that are rapidly developing in recent decades (see, for example, monographs [1-4]). An important class of queueing models is Priority models, models in which requests are endowed with some advantage in service. In this paper we will present some results regarding the analysis of priority models with non-zero switchover time at the service process from one class of priority to another. This time C_{ij} we will call it orientation and we will consider it a random variable with arbitrary distribution function (d.f.) $C_{ij}(x) = P\{C_{ij} < x\}$. Taking into account the orientation time brings about the appearance of new priority laws, more flexible and more advanced compared to classical ones.We'll denote by two indexes IJ the priority laws. The first index will show the future state of interrupted switching, the second - the future state of interrupted service [5]. In this context, we will present some results that can be viewed as multidimensional analogue (in the sense of priority classes) of Kendall functional equation, virtual analogue of Pollaczek-Khintchin equation, etc.

2. System's busy period $M_r|G_r|1|\infty$

Denote by $B_i(x)$-d.f. of service of the requests of the i-th priority class, $C_j(x)$-d.f. of orientation for service of the requests of the j class, λ_i-parameter of the Poisson flow of priority i, $\Pi(x)$-d.f.of the busy period; $i, j = 1, \dots, r$; $i \neq j$, $\sigma_k = \lambda_1 + \dots + \lambda_k$, $\sigma = \sigma_r$, $\beta_i(s) = \int_0^\infty e^{-sx} dB_i(x), c_j(s), \pi(s)$-the Laplace-Stieltjes transform of d.f. $B_i(x)$, $C_j(x)$, $\Pi(x)$.

Statement 1. (Priority policy P12: "resume", "repeat again")
The Laplace-Stieltjes transform $\pi(s) = \pi_r(s)$ of the d.f. of the busy period is determined (at $k = r$) from the system of recurrent functional equations:

$$\sigma_k \pi_k(s) = \sigma_{k-1}\pi_{k-1}(s + \lambda_k) + \sigma_{k-1}\{\pi_{k-1}(s + \lambda_k[1 - \overline{\pi}_k(s)]) -$$

$$-\pi_{k-1}(s + \lambda_k)\}\nu_k(s + \lambda_k[1 - \overline{\pi}_k(s)]) + \lambda_k \pi_{kk}(s), \tag{1}$$

$$\pi_{kk}(s) = \nu_k(s + \lambda_k[1 - \overline{\pi}_k(s)])\overline{\pi}_k(s), \tag{2}$$

$$\overline{\pi}_k(s) = h_k(s + \lambda_k[1 - \overline{\pi}_k(s)]), \tag{3}$$

where

$$\nu_k(s) = c_k(s + \sigma_{k-1}[1 - \pi_{k-1}(s)]), \tag{4}$$

$$h_k(s) = \beta_k(s + \sigma_{k-1}) \left\{ 1 - \frac{\sigma_{k-1}}{s + \sigma_{k-1}} [1 - \beta_k(s + \sigma_{k-1})] \pi_{k-1}(s) \nu_k(s) \right\}^{-1}. \tag{5}$$

Statement 2. (Priority policy P11: "resume", "resume")
The Laplace-Stieltjes transform $\pi(s) = \pi_r(s)$ of the d.f. of the busy period is determined (at $k = r$) from the system of recurrent functional equations:

$$\sigma_k \pi_k(s) = \sigma_{k-1}\pi_{k-1}(s + \lambda_k) + \sigma_{k-1}\{\pi_{k-1}(s + \lambda_k[1 - \overline{\pi}_k(s)]) -$$

$$-\pi_{k-1}(s + \lambda_k)\}\nu_k(s + \lambda_k[1 - \overline{\pi}_k(s)]) + \lambda_k \pi_{kk}(s),$$

$$\pi_{kk}(s) = \nu_k(s + \lambda_k[1 - \overline{\pi}_k(s)])\overline{\pi}_k(s),$$

$$\overline{\pi}_k(s) = h_k(s + \lambda_k[1 - \overline{\pi}_k(s)]),$$

where

$$\nu_k(s) = c_k(s + \sigma_{k-1}[1 - \pi_{k-1}(s)]),$$

$$h_k(s) = \beta_k(s + \Lambda_{k-1}[1 - \pi_{k-1}(s)\nu_k(s)]).$$

Remark 1. *Gnedenko system's busy period.*
If $C_j = 0$, $j = 1, \dots, r$, $r > 1$ from relations (1)-(5) follow the result published by Gnedenko et al in "Priority Queueing Systems" (1973)

$$\sigma_k \pi_k(s) = \sigma_{k-1}\pi_{k-1}(s + \lambda_k(1 - \pi_{kk}(s))) + \lambda_k \pi_{kk}(s),$$

$$\pi_{kk}(s) = h_k(s + \lambda_k(1 - \pi_{kk}(s))),$$

$$h_k(s) = \beta_k(s+\sigma_{k-1})\left\{1 - \frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-\beta_k(s+\sigma_{k-1})]\pi_{k-1}(s)\right\}^{-1}.$$

Remark 2. *Kendall - Takacs equation.*
If $C_j = 0$, $r = 1$ the system (1)-(5) represents a single equation

$$\pi_{11}(s) = h_1(s + \lambda_1(1 - \pi_{11}(s))).$$

But if $r = 1$ result that $h_1(s) = \beta_1(s)$ and $\pi_{11}(s) = \pi(s)$. Considering $\lambda_1 = \lambda$ and $\beta_1 = \beta$ the following equation holds: (known as Kendall - Takacs (1953) functional equation for the busy period for $M|G|1$)

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s))$$

Thus, system (1)-(5) can be considered as n-dimensional analog (n is the number of priority classes) of Kendall - Takacs equation.

3. Steady state condition and traffic coefficients

Let us denote by β_{k1}, c_{i1}, $\pi_{k1}, \ldots, \nu_{k1}$ the first moment of the d.f. $B_k(x), C_k(x), \Pi_k(x), \ldots, N_k(x)$.

Statement 3. (Priority policy P12: "resume", "repeat again")

Let consider $\rho_k = \sum\limits_{i=1}^{k} \lambda_i b_i$,

where

$$b_1 = \frac{\beta_{11} + c_{11}}{1 + \lambda_1 c_{11}},$$

$$b_i = \Phi_1 \cdots \Phi_{i-1} \frac{1}{\sigma_{i-1}}\left[\frac{1}{\beta_i(\sigma_{i-1})} - 1\right](1 + \sigma_{i-1}c_{i1}),$$

$$\Phi_1 = 1, \Phi_i = 1 + (\sigma_i - \sigma_{i-1}\pi_{i-1}(\lambda_i))c_{i1}, \quad i = 2, \ldots, k.$$

If

$$\rho_k < 1 \tag{6}$$

then

$$\sigma_k \pi_{k1} = \frac{\Phi_2 \cdots \Phi_k + \rho_{k-1}}{1 - \rho_k}, \qquad \overline{\pi}_{k1} = \frac{b_k}{1 - \rho_k}, \tag{7}$$

$$h_{k1} = \frac{b_k}{1 - \rho_{k-1}}, \qquad \nu_{k1} = \frac{\Phi_2 \cdots \Phi_{k-1}}{1 - \rho_{k-1}} c_{k1}.$$

Statement 4. (Priority policy P11: "resume", "resume")

Let $\rho_k = \sum\limits_{i=1}^{k} \lambda_i b_i$,

where

$$b_1 = \frac{\beta_{11} + c_{11}}{1 + \lambda_1 c_{11}},$$

$$b_k = \Phi_1 \dots \Phi_{k-1}\beta_{k1}(1+\sigma_{i-1}c_{i1})$$

$$\Phi_1 = 1, \Phi_i = 1 + (\sigma_i - \sigma_{i-1}\pi_{i-1}(\lambda_i))c_{i1}, \quad i = 2, \dots, k.$$

If

$$\rho_k < 1$$

then

$$\sigma_k\pi_{k1} = \frac{\Phi_2\cdots\Phi_k + \rho_{k-1}}{1-\rho_k}, \qquad \overline{\pi}_{k1} = \frac{b_k}{1-\rho_k},$$

$$h_{k1} = \frac{b_k}{1-\rho_{k-1}}, \qquad \nu_{k1} = \frac{\Phi_2\cdots\Phi_{k-1}}{1-\rho_{k-1}}c_{k1}.$$

4. Algorithms

Algorithm P11 (preemptive priority policy: "resume", "resume")
Input: r, s^*, $\epsilon > 0$, $\{\lambda_k\}_{k=1}^r$, $\{\beta_k(s)\}_{k=1}^r$, $\{c_k(s)\}_{k=1}^r$. Output: $\pi_k(s^*)$
Description: *IF* $(k == 0)$ *THEN* $\pi_0(s^*) := 0$;*RETURN* $k := 1$; $q := 1$; $\Lambda_0 := 0$; *Repeat* *inc*(q); $\Lambda_q := \Lambda_{q-1} + \lambda_q$; *Until* $q == r$;
Repeat

$$\nu_k(s) := c_k(s^* + \Lambda_{k-1}[1 - \pi_{k-1}(s^*)]);$$

$$h_k(s^*) := \beta_k(s^* + \Lambda_{k-1}[1 - \pi_{k-1}(s^*)\nu_k(s^*)]);$$

$\pi_{kk}^{(0)}(s^*) := 0$; $n := 1$; *Repeat* $\pi_{kk}^{(n)}(s^*) := h_k(s^* + \lambda_k - \lambda_k\pi_{kk}^{(n-1)}(s^*))$; *inc*$(n)$; *Until* $|\pi_{kk}^{(n)}(s^*) - \pi_{kk}^{(n-1)}(s^*)| < \epsilon$;

$$\pi_k(s^*) := \frac{\Lambda_{k-1}\pi_{k-1}(s^*+\lambda_k)}{\Lambda_k} + \frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s^* + \lambda_k - \lambda_k\pi_{kk}(s^*)) -$$

$$-\pi_{k-1}(s^*+\lambda_k))\nu_k(s^*+\lambda_k[1-\pi_{kk}(s^*)]) + \frac{\lambda_k}{\Lambda_k}\nu(s^*+\lambda_k-\lambda_k\pi_{kk}(s^*))\pi_{kk}(s^*);$$

inc(k); *Until* $k == r$;
End of Algorithm P11.

Algorithm P12 (preemptive priority policy:"resume", "repeat again")
Input: r, s^*, $\epsilon > 0$, $\{\lambda_k\}_{k=1}^r$, $\{\beta_k(s)\}_{k=1}^r$, $\{c_k(s)\}_{k=1}^r$. Output: $\pi_k(s^*)$
Description:*IF (k==0) THEN* $\pi_0(s^*) := 0$; *RETURN* $k := 1$; $q := 1$; $\Lambda_0 := 0$; *Repeat* inc(q);$\Lambda_q := \Lambda_{q-1} + \lambda_q$; *Until* $q == r$;

Repeat $\nu_k(s) := c_k(s^* + \Lambda_{k-1}[1 - \pi_{k-1}(s^*)])$; $h_k(s^*) := \beta_k(s + \Lambda_{k-1})\{1 - \frac{\Lambda_{k-1}}{s^*+\Lambda_{k-1}}[1 - \beta_k(s^*+\Lambda_{k-1})]\pi_{k-1}(s^*)\nu_k(s^*)\}^{-1}$; $\pi_{kk}^{(0)}(s^*) := 0$; $n := 1$; *Repeat* $\pi_{kk}^{(n)}(s^*) := h_k(s^* + \lambda_k - \lambda_k\pi_{kk}^{(n-1)}(s^*))$; *inc*$(n)$; *Until* $|\pi_{kk}^{(n)}(s^*) - \pi_{kk}^{(n-1)}(s^*)| < \epsilon$;

$$\pi_k(s^*) := \frac{\Lambda_{k-1}\pi_{k-1}(s^*+\lambda_k)}{\Lambda_k} + \frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s^*+\lambda_k-\lambda_k\pi_{kk}(s^*)) - \pi_{k-1}(s^*+$$

$$+\lambda_k))\nu_k(s^*+\lambda_k[1-\pi_{kk}(s^*)]) + \frac{\lambda_k}{\Lambda_k}\nu(s^*+\lambda_k-\lambda_k\pi_{kk}(s^*))\pi_{kk}(s^*);$$

$inc(k)$; *Until* $k == r$;
End of Algorithm P12.

5. Conclusions

The stationarity conditions how and traffic coefficients can be applied in the management of real systems to avoid the system's overload. But their application requires solving the systems of functional equations for busy period. The elaborated algorithms solve this problem.

References

1. *Alfa A. S.* Queueing Theory for Telecommunication. — Springer, 2010.
2. *Wuyi Y., Yutake T., Takagi H.* Advances in Queueing Theory and Network. — Springer, 2009.
3. *Vishnevsky V. V.* Theoretical bases of design of the communication network. — Moscow, Tehnosphera, 2003.
4. *Vishnevsky V. V., Semenova O. V.,* Polling Systems: Theory and Applications for Broadband Wireless Networks. — London, Academic Publishing, 2012.
5. *Mishkoy Gh. K.* Generalized Priority Systems. — Chisinau, Academy of Sciences of Moldova, 2009 (in Russian).