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# On a multi-server priority queue with preemption in crowdsourcing

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**Abstract.** A $c-$server queueing system providing service to two types of customers, say, Type 1 and Type 2 to which customers arrive according to a marked Poisson process is considered. A Type 1 customer has to receive service by one of $c$ servers while a Type 2 customer may be served by a Type 1 customer (with probability $p$) who is available to act as a server soon after getting own service or by one of $c$ servers. Upon completion of a service a free server will offer service to a Type 1 customer on a FCFS basis. However, if there is no Type 1 customer waiting in the system, that server will serve a Type 2 customer if one of that type is present in the queue. The service time is exponentially distributed for each category. We consider preemptive service discipline. Condition for system stability is established. Crucial system characteristics are computed.

**Keywords:** crowdsourcing, queueing system, preemptive service, matrix-analytic method.

## 1. Introduction

In this paper we analyze the impact of preemptive priority in the context of crowdsourcing. For a detailed discussion on crowdsourcing one may refer to Chakravarthy and Dudin [2]. That is the first reported work on crowdsourcing modelled in the queueing context. They analyze the problem as a priority queue with non-preemption.

Priority queues have been extensively investigated by several researches (see Brodal [1], Jaiswal ( [4], [5]), Takagi [10]). Brodal [1] provides a survey on priority queues in the context of binary heap. Book by Takagi [10] gives a detailed account of development of priority queues up to the early 1990's. These are broadly classified into preemptive and non-preemptive disciplines. In non-preemptive priority discipline the service of a customer of lower priority is not affected by the arrival of a customer of higher priority. However, in preemptive discipline the lower priority customer in service is instantly replaced by the higher priority on its arrival. The latter case can be regarded as one of service interruption. An extensive survey on queues with interruption is provided in Krishnamoorthy et al. [6].

We consider a multi-server priority model in the context of crowdsourcing with two types of customers – Type 1 and Type 2 to which customers arrive according to Poisson process of rates $\lambda_1$ and $\lambda_2$ respectively. Type 1 has priority over Type 2, which is of preemptive nature. Type 1 and Type 2 customers are to be served by one of $c$ servers and the service times are assumed to be exponential with respective parameters $\mu_1$ and $\mu_2$. Services are offered in the order of the arrivals of the customers. Type 2 customers may be served by a Type 1 customer who has been served out and also available to act as a server immediately after his service completion. At the time of opting to serve there should be at least one Type 2 customer waiting to get a service. We assume that a served Type 1 customer will be available to serve a waiting Type 2 customer with probability $p$, $0 \leq p \leq 1$. With probability $q = 1 - p$, the served Type 1 customer will leave the system. If a Type 1 customer decides to serve a Type 2 customer, for our analysis purposes that Type 2 customer will be removed from the system immediately. This is due to the fact that the system no longer needs to track that Type 2 customer. Type 2 customers are taken for service one at a time from the head of the queue whenever the queue of Type 1 customers are found to be empty at a service completion epoch. The service of such customers is according to a preemptive service discipline, that is the arrival of a Type 1 customer interrupts the ongoing service of any one of Type 2 customers if any in service, and hence this preempted customer joins back as the head of the Type 2 queue. Type 1 customers have a limited waiting space $L$, $1 \leq L < \infty$, while Type 2 customers have unlimited waiting space (see Figure 1). The above described service is found in what is referred to as “crowdsourcing" (see Howe [3]). A typical example for our model is supermarkets where customers (Type 1) arrive physically and get served, whereas Type 2 customers are those who place order online or over phone.

The present paper differs from Chakravarthy and Dudin [2] mainly in the fact that the former is on preemptive priority discipline. Thus several of system performance measures in the two cases differ significantly. Even the stability condition differ significantly in the two cases.

The rest of the paper is arranged as follows. In Section 2 the model under study is described. Section 3 provides the steady state analysis of the model, including key performance measures. Numerical illustrations are presented in Section 4. The following notations are used in the sequel.

- $\mathbf{e}$ : a column vector of 1's of appropriate order
- $I$ : identity matrix of appropriate order

## 2. Mathematical formulation

Let $N_1(t), S(t)$ and $N_2(t)$ be the number of Type 1 customers in the system, the number of servers busy with Type 2 customers and the number of Type 2 customers in the queue respectively.

Then $\Omega = \{(N_2(t), S(t), N_1(t)), t \geq 0\}$ is a continuous time Markov chain with state space

$$\{(0,0,k), 0 \leq k \leq c+L\} \bigcup \{(i,0,k), i \geq 1, c \leq k \leq c+L\} \bigcup$$

$$\{(0,j,k), 1 \leq j \leq c, 0 \leq k \leq c-j\} \bigcup \{(i,j,k), i \geq 1, 1 \leq j \leq c, k = c-j\}.$$

Now we define the set of states as follows:

$$\hat{\mathbf{0}} = \{(0,0,k), 0 \leq k \leq c+L\} \bigcup \{(0,j,k), 1 \leq j \leq c, 0 \leq k \leq c-j\}$$

and for $i \geq 1$,

$$\hat{i} = \{(i,0,k), i \geq 1, c \leq k \leq c+L\} \bigcup \{(i,j,k), i \geq 1, 1 \leq j \leq c, k = c-j\}.$$

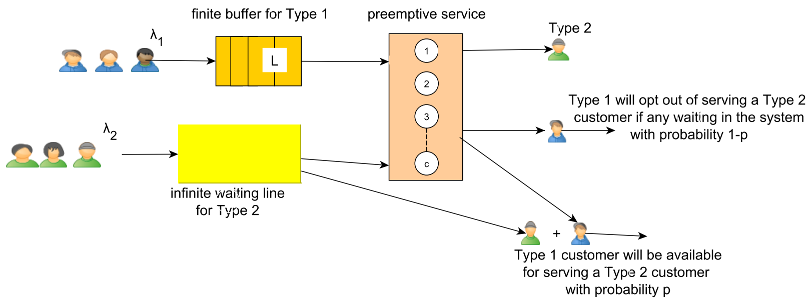


Figure 1. Pictorial description of the model

Note that the level $\hat{\mathbf{0}}$ has $c + L + 1 + \frac{c(c+1)}{2}$ states while the level $\hat{i}, i \geq 1$ has $c+L+1$ states. This continuous time Markov chain is a level independent quasi birth and death process and the infinitesimal generator is

$$\mathcal{Q} = \begin{pmatrix} B_1 & B_0 & & & & \\ B_2 & A_1 & A_0 & & & \\ B_3 & A_2 & A_1 & A_0 & & \\ & A_3 & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Rearranging the generator $\mathcal{Q}$ given in (2) by combining the set of states as

$$\widetilde{i} = \{2\hat{i} - 1, 2\hat{i}\}, \quad i \geq 1,$$

the model under study can be studied as $QBD$ process with the generator $\tilde{\mathcal{Q}}$ is of the form

$$\tilde{\mathcal{Q}} = \begin{pmatrix} B_1 & \tilde{B}_0 & & & \\ \tilde{B}_2 & \tilde{A}_1 & \tilde{A}_0 & & \\ & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$\tilde{B}_0 = \begin{pmatrix} B_0 & 0 \end{pmatrix}, \tilde{B}_2 = \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}, \tilde{A}_0 = \begin{pmatrix} 0 & 0 \\ A_0 & 0 \end{pmatrix}$$

,

$$\tilde{A}_1 = \begin{pmatrix} A_1 & A_0 \\ A_2 & A_1 \end{pmatrix}, \tilde{A}_2 = \begin{pmatrix} A_3 & A_2 \\ 0 & A_3 \end{pmatrix}.$$

## 2.1. Stability condition

Next we examine the system stability. Define $\tilde{A} = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2$. Then

$$\tilde{A} = \begin{pmatrix} A_1 + A_3 & A_0 + A_2 \\ A_0 + A_2 & A_1 + A_3 \end{pmatrix}$$

This is the infinitesimal generator of the finite state continuous time Markov chain. Let $\boldsymbol{\eta}$ be the steady state probability vector of $\tilde{A}$. Then

$$\boldsymbol{\eta}\tilde{A} = 0, \quad \boldsymbol{\eta}\mathbf{e} = 1.$$

Note that $\tilde{A}$ is a circulant matrix, the vector $\boldsymbol{\eta}$ is of the form

$$\boldsymbol{\eta} = \begin{pmatrix} \frac{\boldsymbol{\pi}}{2} & \frac{\boldsymbol{\pi}}{2} \end{pmatrix}$$

where $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi}A = 0, \quad \boldsymbol{\pi}\mathbf{e} = 1$$

with $A = A_0 + A_1 + A_2 + A_3$.

This leads to

$$\pi_i(j) = \begin{cases} \frac{1}{j!}\left(\frac{\lambda_1}{\mu_1}\right)^j \pi_c(0) & 0 \le i \le c-1, \quad j = c-i \\ \left(\frac{1}{c}\right)^{j-c} \frac{1}{c!}\left(\frac{\lambda_1}{\mu_1}\right)^j \pi_c(0) & i = 0, \qquad\qquad c+1 \le j \le c+L \end{cases}$$

where

$$\pi_c(0) = \left[1 + \sum_{i=1}^{c-1} \frac{1}{i!}\left(\frac{\lambda_1}{\mu_1}\right)^i + \frac{1}{c!}\left(\frac{\lambda_1}{\mu_1}\right)^c \sum_{i=0}^{L}\left(\frac{\lambda_1}{c\mu_1}\right)^i\right]^{-1}.$$

The following theorem provides the stability condition of the queueing system under study.

**Theorem 2.1.** *The system under study is stable if and only if*

$$\lambda_2 - p\lambda_1 a < cp\mu_1 a_1 + \mu_2 a_2$$

*where*

$$a = \sum_{i=0}^{c-1} \frac{1}{i!}\left(\frac{\lambda_1}{\mu_1}\right)^i \pi_c(0),$$
$$a_1 = \frac{1}{c!}\left(\frac{\lambda_1}{\mu_1}\right)^c \sum_{i=1}^{L}\left(\frac{\lambda_1}{c\mu_1}\right)^i \pi_c(0),$$
$$a_2 = \sum_{i=0}^{c-1} \frac{c-i}{i!}\left(\frac{\lambda_1}{\mu_1}\right)^i \pi_c(0)$$

*with $\pi_c(0)$ as given above.*

**Theorem 2.2.** *In the case of a single server, the queueing system under study is stable if and only if the following condition is satisfied*

$$\lambda_2 < p\mu_1 + (\mu_2 - p\mu_1)\left(1 - \frac{\lambda_1}{\mu_1}\right)\left[1 - \left(\frac{\lambda_1}{\mu_1}\right)^{L+2}\right]^{-1}.$$

**Remark**: Under the assumption that $\lambda_1 < \mu_1$, when L goes to $\infty$,

$$\pi_1(0) \to \left(1 - \frac{\lambda_1}{\mu_1}\right).$$

Then the stability condition reads as

$$\lambda_2 - p\lambda_1 < \mu_2\left(1 - \frac{\lambda_1}{\mu_1}\right).$$

## 3. Steady state analysis

Let $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, ...)$ be the steady state probability vector of the generator $\tilde{\mathcal{Q}}$. That is,

$$\mathbf{y}\tilde{\mathcal{Q}} = 0, \text{ and } \mathbf{ye} = 1.$$

Note that $\mathbf{y}_0 = \mathbf{x}_0$ and $\mathbf{y}_i = (\mathbf{x}_{2i-1}, \mathbf{x}_{2i})$ for $i \geq 1$ where $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ is the steady state probability vector of $\mathcal{Q}$.

The vectors are partitioned as

$$\mathbf{x}_0 = \{x_0(0,k), 0 \leq k \leq c+L\} \bigcup \{x_0(j,k), 1 \leq j \leq c, 0 \leq k \leq c-j\}$$

and

$$\mathbf{x}_i = \{x_i(0,k), c \leq k \leq c+L\} \bigcup \{x_i(j,k), 1 \leq j \leq c, k = c-j\} \text{ for } i \geq 1.$$

Under the stability condition given in theorem 2.1 the steady-state probability vector $\mathbf{y}$ is obtained as

$$\mathbf{y}_i = \mathbf{y}_1 R^{i-1}, \quad i \geq 2$$

where $R$ is the minimal non-negative solution to the matrix quadratic equation (see Neuts [9] and Latouche and Ramaswami [8])

$$R^2 \tilde{A}_2 + R\tilde{A}_1 + \tilde{A}_0 = \mathbf{0}$$

and the boundary equations are given by

$$\begin{pmatrix} \mathbf{y}_0 & \mathbf{y}_1 \end{pmatrix} \begin{pmatrix} \tilde{B}_1 & \tilde{B}_0 \\ \tilde{B}_2 & \tilde{A}_1 + R\tilde{A}_2 \end{pmatrix} = \mathbf{0}.$$

The normalizing condition results in

$$\mathbf{y}_0\mathbf{e} + \mathbf{y}_1(I-R)^{-1}\mathbf{e} = 1.$$

The matrix $R$ is calculated as

$$R = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ R_1 & R_2 \end{pmatrix}.$$

Define the $(c+L+1)$-dimensional vector $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} = \sum_{i=1}^{\infty} \mathbf{y}_i\mathbf{e} = \mathbf{y}_1(I-R)^{-1}(\mathbf{e} \otimes I) = \mathbf{x}_1 + \mathbf{x}_2(I-R_2)^{-1}(I+R_1).$$

Partition $\boldsymbol{\xi} = (\boldsymbol{\xi}_0, \xi_1, ..., \xi_c)$ as

$$\boldsymbol{\xi}_0 = (\xi(0,c), \xi(0,c+1), ..., \xi(0,c+L))$$

and

$$\xi_j = \xi(j, c-j), \quad 1 \leq j \leq c.$$

Note that $\xi(j,k)$ gives the steady state probability that $j$ servers are busy with Type 2 customers and there are $k$ Type 1 customers in the system.

### 3.1. System Performance Measures

1. Probability that the system is idle is

$$P_{idle} = x_0(0,0)$$

2. Probability that $j$ servers are busy is

$$b_j = \begin{cases} x_0(0,0) & j = 0 \\ \sum_{k=0}^{j} x_0(k, j-k) & 1 \leq j \leq c-1 \\ \sum_{i=0}^{\infty} \left[ \sum_{k=c}^{c+L} x_i(0,k) + \sum_{k=1}^{c} x_i(k, c-k) \right] & j = c \end{cases}$$

3. Probability that $j$ servers are busy with Type 1 customers is

$$b_j^{(1)} = \begin{cases} x_0(0,0) + \sum_{k=1}^{c} x_0(k,0) + \sum_{i=1}^{\infty} x_i(c,0) & j = 0 \\ x_0(0,j) + \sum_{k=1}^{c-j} x_0(k,j) + \sum_{i=1}^{\infty} x_i(c-j,j) & 1 \leq j \leq c-1 \\ \sum_{i=0}^{\infty} \sum_{k=c}^{c+L} x_i(0,k) & j = c \end{cases}$$

4. Probability that $j$ servers are busy with Type 2 customers is given by

$$b_j^{(2)} = \begin{cases} \sum_{k=0}^{c+L} x_0(0,k) + \sum_{i=1}^{\infty} \sum_{k=c}^{c+L} x_i(0,k) & j = 0 \\ \sum_{k=0}^{c-j} x_0(j,k) + \sum_{i=1}^{\infty} x_i(j, c-j) & 1 \leq j \leq c \end{cases}$$

5. Probability that an arriving customer is lost due to lack of buffer space is

$$P_{lost} = x_0(0, c+L) + \xi(0, c+L)$$

6. Mean number of Type 1 customers in the queue is

$$\mu_{N_1} = \sum_{i=0}^{\infty} \sum_{k=c+1}^{c+L} (k-c) x_i(0,k)$$

7. Mean number of Type 2 customers in the queue is

$$\mu_{N_2} = \sum_{i=1}^{\infty} i \left[ \sum_{k=c}^{c+L} x_i(0,k) + \sum_{j=1}^{c} x_i(j, c-j) \right]$$

8. Rate of Type 2 customers leaving with Type 1 customers denoted by $R_{T_2 \to T_1}$ upon service completion of Type 1 customers is

$$R_{T_2 \to T_1} = p\mu_1 \sum_{i=1}^{\infty} \left[ \sum_{k=c}^{c+L} c x_i(0,k) + \sum_{j=1}^{c-1} (c-j) x_i(j, c-j) \right]$$

9. Rate of Type 2 customers leaving the system denoted by $R_{T_2 \to S}$ upon getting service by one of $c-$servers is

$$R_{T_2 \to S} = \mu_2 \left[ \sum_{j=1}^{c} \sum_{k=0}^{c-j} j x_0(j,k) + \sum_{i=1}^{\infty} \sum_{j=1}^{c} j x_i(j, c-j) \right]$$

10. Rate of Type 2 customers preempted by Type 1 customers is

$$R_{T_2 \to P} = \lambda_1 \sum_{i=0}^{\infty} \sum_{j=1}^{c} x_i(j, c-j)$$

11. Probability that Type 2 customers leaving with Type 1 customers upon service completion of Type 1 customers is

$$\mathcal{P}_{T_2 \to T_1} = \frac{p\mu_1}{\lambda_2} \left[ \sum_{k=c}^{c+L} c\xi(0,k) + \sum_{j=1}^{c} (c-j)\xi(j, c-j) \right]$$

12. Probability that Type 2 customers leaving the system upon getting service by one of $c-$servers is

$$\mathcal{P}_{T_2 \to S} = \frac{\mu_2}{\lambda_2} \sum_{j=1}^{c} \left[ \sum_{k=0}^{c-j} j x_0(j,k) + j\xi(j, c-j) \right]$$

13. Probability that Type 2 customers preempted by Type 1 customers is

$$\mathcal{P}_{T_2 \to P} = \frac{\lambda_1}{\lambda_2} \sum_{j=1}^{c} \left[ x_0(j, c-j) + \xi(j, c-j) \right]$$

### 3.2. Waiting time of an Admitted Type 1 customer in the queue

For computing expected waiting time of an admitted Type 1 customer in the queue, we consider the Markov chain $\{M(t), t \geq 0\}$ where $M(t)$ is the position of the admitted customer in the queue. We arrange the state space as $\{1, 2, ..., L\} \bigcup \{\Delta\}$ where $\{\Delta\}$ is the absorbing state denoting the admitted Type 1 customer taken for service. Thus the infinitesimal generator is of the form

$$W = \begin{pmatrix} T & \mathbf{T}^0 \\ \mathbf{0} & 0 \end{pmatrix}$$

where

$$T = \begin{pmatrix} -c\mu_1 & & & \\ c\mu_1 & -c\mu_1 & & \\ & \ddots & \ddots & \\ & & c\mu_1 & -c\mu_1 \end{pmatrix}, \mathbf{T}^0 = \begin{pmatrix} c\mu_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus waiting time of an admitted Type 1 customer at an arrival epoch follows a Phase type distribution with representation $(\boldsymbol{\alpha}, T)$ of order $L$ with the initial probability vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_L)$ where

$$\alpha_j = \frac{1}{1 - P_{lost}}(x_0(0, c + j - 1) + \xi(0, c + j - 1).$$

That is, $\alpha_j$, $1 \leq j \leq L$ is the probability that an admitted Type 1 customer finds $(j - 1)$ Type 1 customers waiting in the queue with $c$ servers busy with Type 1 customers. Since Type 1 customers have preemptive priority over Type 2 customers, there is no need to keep track of the number of Type 2 customers in the queue and future arrival of any type.

Expected waiting time in the queue of an admitted Type 1 customer is given by

$$\mu_W^{(1)} = -\boldsymbol{\alpha} T^{(-1)} \mathbf{e} = \frac{1}{c\mu_1}(\alpha_1 + 2\alpha_2 + ... + L\alpha_L).$$

## 4. Numerical illustrations

In this section we discuss a few numerical examples. In the following we define $\rho$ as

$$\rho = \frac{\lambda_2}{p\lambda_1 a + cp\mu_1 a_1 + \mu_2 a_2}.$$

Whenever we need to fix a specific value for $\rho$, we can vary any of the system parameters $\lambda_1, \mu_1, \mu_2, L, c$ and $p$ to arrive at that value. However, $a, a_1, a_2$ and the vector $\boldsymbol{\pi}$ are independent of $\lambda_2$. Thus, for a specific value of $\rho$ from (4) we have $\lambda_2$.

**Example 1.**

In this example we consider the behaviour of the measure $\mathcal{P}_{T_2 \to T_1}$. We fix $\lambda_1 = 1, \mu_1 = \mu_2 = 1.1$, vary $p$ to take values 0.5 and 1, $c$ from 1 to 3 and $\rho$ take values 0.1, 0.3, 0.5, 0.7, 0.9, 0.95 and 0.99 (see Table 1).

Table 1 gives a picture of the behaviour of $\mathcal{P}_{T_2 \to T_1}$ for $p = 0.5$ and 1 and with $\rho$ varying from 0.1 to 0.99. We notice that the fraction $\mathcal{P}_{T_2 \to T_1}$ decreases with increasing value of $\rho$; in the case of single server and for fixed $\rho$, the fraction keeps increasing with increasing value of $L$. The latter behaviour is seen to be exhibited for the multi-server case also. However, when number of servers is 3 or more the fraction $\mathcal{P}_{T_2 \to T_1}$ increases with increasing value of $\rho$. This is so since more and more Type 1 customers get admitted to the system. However, for small values of $\lambda_1$, we notice that increase in value of $c$ results in more and more Type 2 customers getting served in the absence of Type 1 customers. This explains the reason for small values for $\mathcal{P}_{T_2 \to T_1}$ for $c = 3$.

**Example 2.**

Table 2 we investigate the behavior of $\lambda_2$ at which the measure $\mathcal{P}_{T_2 \to T_1}$ attains its maximum. Fix $\lambda_1 = 1, \mu_1 = \mu_2 = 1.1$, vary $p$ to 0.1, 0.2, 0.5, 0.8 and 1, $c$ from 1 to 5, vary $L$ to be 5, 10 and 15. First we get the value of $\rho$ at which $\mathcal{P}_{T_2 \to T_1}$ attains its maximum then we obtain corresponding value of $\lambda_2$.

**Example 3.**

In Table 3 we compute the optimum value of $L$, say $L^*$ and value of $\lambda_2$ at $L^*$. The optimum $L^*$ is such that the system measure $P_{lost}$ is no larger than $10^{-4}$ when all other parameters are fixed. We fix $\lambda_1 = 1, \mu_1 = \mu_2 = 1.1$, vary $p$ to 0, 0.5, 1, $c$ from 1 to 5 and $\rho$ take values 0.1, 0.3, 0.5, 0.8, 0.9 and 0.95.

Table 3 reveals certain interesting observation: for small values of $\rho$ (hence small values of $\lambda_2$ the optimal value of $L$ is relatively small compared to moderate to high values of $L$ for larger values of $\rho$ (hence large values of $\lambda_2$).

**Revenue function**

Define revenue function as

$$\mathcal{R}_f(\mu_1) = \mathcal{C}_1\ R_{T_2 \to T_1} - \mathcal{C}_2\ R_{T_2 \to P} - \mathcal{C}_3\ P_{lost} - \mathcal{C}_4\ \mu_{N_2}$$

where

- $\mathcal{C}_1$ : Revenue to the system on account of a waiting Type 2 customer, served by a departing Type 1 customer
- $\mathcal{C}_2$ : Preemption cost per unit Type 2 customer
- $\mathcal{C}_3$ : Cost of a Type 1 customer lost due to finite waiting space
- $\mathcal{C}_4$ : Holding cost per Type 2 customer

In order to study the variation in $\mu_1$ on profit/ revenue function we fix the costs $\mathcal{C}_1 = \$50, \mathcal{C}_2 = \$10, \mathcal{C}_3 = \$15, \mathcal{C}_4 = \$5$.

For this profit function we get output as indicated in Table tab:4. There is an indication for this profit function to have a global optimum. In the present case the optimal service rate for Type 1 customers turn out to be $\mu_1 = 12$. Values of $\mu_1$ above 12 result in very high preemption cost, whereas those below 12 result in large number of Type 1 customers loss.

## 5. Conclusions

The main advantage with the problem we analyzed in this paper in comparison with that of Chakravarthy and Dudin [2] is that the loss of Type 1 customers is reduced due to preemption. The rate of loss of Type 1 in our case is probability of $c+L$ Type 1 customers in the system whereas that in Chakravarthy and Dudin [2] is

$\sum_{i=0}^{c} Prob.(c-i$ Type 1 in service and $L$in waiting ). This results in a larger number of Type 2 customers being served by Type 1 customers. However, preemption of a Type 2, sometimes even more than once, may lead to its longer waiting time in the system. Nevertheless if suitable incentives is provided to Type 1 customers who serve a Type 2 customers on leaving the system, then $p$ may become close to 1, if not equal to 1.

In a future work we propose to consider the extension of the present model to queueing-inventory scenario.

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## References

1. *Brodal G. S.* A Survey on Priority Queues: Lecture Notes in Computer Science. —2013— Vol. 8066 P. 150–163.
2. *Chakravarthy S. R. ,Dudin A. N.* A Queueing Model for Crowdsourcing. —Journal of the Operational Research Society —2016.
3. *Howe J.* Crowdsourcing: A definition. —URL:http://www.crowdsourcing.com/ cs/2006/06/ crowdsourcing a.html.[Online] 2006.
4. *Jaiswal N. K.* Preemptive resume priority queue. —Operations Research —1961 —No. 9. P. 732–742.

5. *Jaiswal N. K.* Priority queues. —Academic Press —1968 —New York and London.
6. *Krishnamoorthy A.,Pramod P. K., Chakravarthy S. R.* Queues with interruptions, a survey // Top. —2014. —22(1). —P. 290-–320.
7. *Latouche G., Ramaswami V.* A logarithmic reduction algorithm for quasi-birth-and-death processes // Journal of Applied Probability. —1993. —30. P. 650–674 .
8. *Latouche G., Ramaswami V.* An Introduction to Matrix Analytic Methods in Stochastic Modeling // SIAM. —1999. —Philadelphia.
9. *Neuts M. F* Matrix-geometric solutions in stochastic models: an algorithmic approach. — The Johns Hopkins University Press, Baltimore [1994 version is Dover Edition]. — 1981.
10. *Takagi H.* Queueing Analysis: Vacations and Priority Systems. — 1991. — North- Holland: Amsterdam. — Vol. 1.

| $c$ | $p$ | $\rho$ | $L = 5$ | $L = 10$ | $L = 15$ | $L = 20$ | $L = 25$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.1 | 0.8352 | 0.8828 | 0.9005 | 0.909 | 0.9136 |
| | | 0.3 | 0.8185 | 0.8729 | 0.8923 | 0.9016 | 0.9066 |
| | | 0.5 | 0.7855 | 0.8492 | 0.8710 | 0.8813 | 0.8872 |
| | | 0.7 | 0.7243 | 0.7920 | 0.8154 | 0.8276 | 0.8358 |
| | | 0.9 | 0.6464 | 0.7129 | 0.7378 | 0.7523 | 0.7629 |
| | | 0.95 | 0.6267 | 0.6927 | 0.7180 | 0.7329 | 0.7438 |
| | | 0.99 | 0.6112 | 0.6768 | 0.7024 | 0.7176 | 0.7286 |
| | 1 | 0.1 | 0.8995 | 0.9282 | 0.9390 | 0.9442 | 0.9470 |
| | | 0.3 | 0.8927 | 0.9233 | 0.9348 | 0.9404 | 0.9434 |
| | | 0.5 | 0.8773 | 0.9103 | 0.9224 | 0.9284 | 0.9320 |
| | | 0.7 | 0.8336 | 0.8675 | 0.8805 | 0.8885 | 0.8944 |
| | | 0.9 | 0.7605 | 0.7948 | 0.8101 | 0.8208 | 0.8290 |
| | | 0.95 | 0.7402 | 0.7749 | 0.7907 | 0.8017 | 0.8101 |
| | | 0.99 | 0.7239 | 0.7590 | 0.7750 | 0.7862 | 0.7946 |
| 2 | 0.5 | 0.1 | 0.3648 | 0.3664 | 0.3664 | 0.3664 | 0.3664 |
| | | 0.3 | 0.3589 | 0.3606 | 0.3607 | 0.3607 | 0.3607 |
| | | 0.5 | 0.3499 | 0.3519 | 0.3522 | 0.3524 | 0.3527 |
| | | 0.7 | 0.3379 | 0.3412 | 0.3431 | 0.3451 | 0.3473 |
| | | 0.9 | 0.3236 | 0.3306 | 0.3369 | 0.3436 | 0.3505 |
| | | 0.95 | 0.3198 | 0.3281 | 0.3360 | 0.3443 | 0.3527 |
| | | 0.99 | 0.3167 | 0.3262 | 0.3354 | 0.3450 | 0.3549 |
| | 1 | 0.1 | 0.4815 | 0.4828 | 0.4829 | 0.4829 | 0.4829 |
| | | 0.3 | 0.4880 | 0.4893 | 0.4893 | 0.4893 | 0.4893 |
| | | 0.5 | 0.4899 | 0.4914 | 0.4916 | 0.4917 | 0.4919 |
| | | 0.7 | 0.4853 | 0.4883 | 0.4904 | 0.4924 | 0.4946 |
| | | 0.9 | 0.4735 | 0.4825 | 0.4911 | 0.4997 | 0.5082 |
| | | 0.95 | 0.4696 | 0.4810 | 0.4921 | 0.5031 | 0.5139 |
| | | 0.99 | 0.4662 | 0.4798 | 0.4931 | 0.5062 | 0.5190 |
| 3 | 0.5 | 0.1 | 0.1261 | 0.1262 | 0.1262 | 0.1262 | 0.1262 |
| | | 0.3 | 0.1345 | 0.1346 | 0.1346 | 0.1346 | 0.1346 |
| | | 0.5 | 0.1425 | 0.1426 | 0.1426 | 0.1426 | 0.1426 |
| | | 0.7 | 0.1498 | 0.1499 | 0.1500 | 0.1501 | 0.1502 |
| | | 0.9 | 0.1563 | 0.1567 | 0.1571 | 0.1576 | 0.1580 |
| | | 0.95 | 0.1577 | 0.1583 | 0.1589 | 0.1595 | 0.1602 |
| | | 0.99 | 0.1589 | 0.1596 | 0.1603 | 0.1611 | 0.1620 |
| | 1 | 0.1 | 0.1897 | 0.1898 | 0.1898 | 0.1898 | 0.1898 |
| | | 0.3 | 0.2111 | 0.2112 | 0.2112 | 0.2112 | 0.2112 |
| | | 0.5 | 0.2310 | 0.2311 | 0.2311 | 0.2311 | 0.2311 |
| | | 0.7 | 0.2487 | 0.2489 | 0.2490 | 0.2491 | 0.2493 |
| | | 0.9 | 0.2636 | 0.2644 | 0.2653 | 0.2662 | 0.2671 |
| | | 0.95 | 0.2669 | 0.2681 | 0.2693 | 0.2706 | 0.2719 |
| | | 0.99 | 0.2694 | 0.2709 | 0.2725 | 0.2742 | 0.2759 |

Table 1

Effect of $c, p, \rho, L$ on $\mathcal{P}_{T_2 \to T_1}$

| $c$ | $p$ | $L = 5$ | | $L = 10$ | | $L = 15$ | |
|---|---|---|---|---|---|---|---|
| | | $\mathcal{P}_{T_2 \to T_1}$ | $\lambda_2$ | $\mathcal{P}_{T_2 \to T_1}$ | $\lambda_2$ | $\mathcal{P}_{T_2 \to T_1}$ | $\lambda_2$ |
| 1 | 0.1 | 0.2715 | 0.2919 | 0.3543 | 0.2397 | 0.3971 | 0.2200 |
| | 0.2 | 0.4160 | 0.3805 | 0.5066 | 0.3340 | 0.5477 | 0.3165 |
| | 0.5 | 0.6112 | 0.6462 | 0.6768 | 0.6171 | 0.7024 | 0.6062 |
| | 0.8 | 0.6922 | 0.9119 | 0.7368 | 0.9003 | 0.7553 | 0.8959 |
| | 1 | 0.7239 | 1.0890 | 0.7590 | 1.0890 | 0.7750 | 1.0890 |
| 2 | 0.1 | 0.0905 | 0.7432 | 0.0931 | 0.7404 | 0.0952 | 0.7404 |
| | 0.2 | 0.1631 | 0.8416 | 0.1678 | 0.8394 | 0.1721 | 0.8394 |
| | 0.5 | 0.3167 | 1.1366 | 0.3262 | 1.1364 | 0.3354 | 1.1364 |
| | 0.8 | 0.4165 | 1.4317 | 0.4288 | 1.4333 | 0.4409 | 1.4334 |
| | 1 | 0.4662 | 1.6284 | 0.4798 | 1.6313 | 0.4931 | 1.6314 |
| 3 | 0.1 | 0.0382 | 0.8018 | 0.0383 | 0.8016 | 0.0385 | 0.8016 |
| | 0.2 | 0.0724 | 0.9007 | 0.0727 | 0.9006 | 0.0730 | 0.9006 |
| | 0.5 | 0.1589 | 1.1974 | 0.1596 | 1.1976 | 0.1603 | 1.1976 |
| | 0.8 | 0.2290 | 1.4940 | 0.2302 | 1.4946 | 0.2314 | 1.4946 |
| | 1 | 0.2694 | 1.6918 | 0.2709 | 1.6926 | 0.2725 | 1.6926 |
| 4 | 0.1 | 0.0122 | 0.6398 | 0.0122 | 0.6398 | 0.0122 | 0.6398 |
| | 0.2 | 0.0244 | 0.7387 | 0.0244 | 0.7388 | 0.0244 | 0.7388 |
| | 0.5 | 0.0615 | 1.0356 | 0.0615 | 1.0358 | 0.0615 | 1.0358 |
| | 0.8 | 0.0996 | 1.3325 | 0.0996 | 1.3328 | 0.0996 | 1.3328 |
| | 1 | 0.1251 | 1.5305 | 0.1252 | 1.5308 | 0.1253 | 1.5308 |
| 5 | 0.1 | 0.0028 | 0.4666 | 0.0028 | 0.4666 | 0.0028 | 0.4666 |
| | 0.2 | 0.0059 | 0.5656 | 0.0059 | 0.5656 | 0.0059 | 0.5656 |
| | 0.5 | 0.0176 | 0.8626 | 0.0176 | 0.8626 | 0.0176 | 0.8626 |
| | 0.8 | 0.03300 | 1.1596 | 0.0330 | 1.1596 | 0.0330 | 1.1596 |
| | 1 | 0.0452 | 1.3575 | 0.0452 | 1.3576 | 0.0452 | 1.3576 |

Table 2

Value of $\lambda_2$ at which $\mathcal{P}_{T_2 \to T_1}$ attains its maximum

| $\rho$ | $p$ | $c=1$ | | $c=2$ | | $c=3$ | | $c=4$ | | $c=5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $L$ | $\lambda_2$ | $L$ | $\lambda_2$ | $L$ | $\lambda_2$ | $L$ | $\lambda_2$ | $L$ | $\lambda_2$ |
| 0.1 | 0 | 47 | 0.0101 | 7 | 0.0649 | 4 | 0.0711 | 2 | 0.0551 | 1 | 0.0381 |
| | 0.5 | 47 | 0.0600 | 7 | 0.1148 | 4 | 0.1209 | 2 | 0.1036 | 1 | 0.0821 |
| | 1 | 47 | 0.1100 | 7 | 0.1647 | 4 | 0.1707 | 2 | 0.1522 | 1 | 0.1261 |
| 0.3 | 0 | 48 | 0.0303 | 7 | 0.1946 | 4 | 0.2133 | 2 | 0.1653 | 1 | 0.1144 |
| | 0.5 | 48 | 0.1801 | 7 | 0.3444 | 4 | 0.3627 | 2 | 0.3109 | 1 | 0.2463 |
| | 1 | 48 | 0.3300 | 7 | 0.4942 | 4 | 0.5121 | 2 | 0.4565 | 1 | 0.3783 |
| 0.5 | 0 | 51 | 0.0503 | 7 | 0.3243 | 4 | 0.3554 | 2 | 0.2754 | 1 | 0.1906 |
| | 0.5 | 60 | 0.3001 | 7 | 0.5740 | 4 | 0.6045 | 2 | 0.5182 | 1 | 0.4105 |
| | 1 | 74 | 0.5500 | 7 | 0.8236 | 4 | 0.8535 | 2 | 0.7609 | 1 | 0.6304 |
| 0.8 | 0 | 163 | 0.0800 | 8 | 0.5185 | 4 | 0.5687 | 2 | 0.4407 | 1 | 0.3050 |
| | 0.5 | 383 | 0.4800 | 8 | 0.9183 | 4 | 0.9672 | 2 | 0.8291 | 1 | 0.6569 |
| | 1 | 425 | 0.8800 | 9 | 1.3182 | 4 | 1.3657 | 2 | 1.2174 | 1 | 1.0087 |
| 0.9 | 0 | 169 | 0.0900 | 8 | 0.5834 | 4 | 0.6398 | 2 | 0.4958 | 1 | 0.3432 |
| | 0.5 | 483 | 0.5400 | 10 | 1.0331 | 4 | 1.0881 | 2 | 0.9327 | 1 | 0.7390 |
| | 1 | 537 | 0.9900 | 16 | 1.4831 | 4 | 1.5364 | 2 | 1.3696 | 1 | 1.1348 |
| 0.95 | 0 | 171 | 0.0950 | 8 | 0.6158 | 4 | 0.6753 | 2 | 0.5234 | 1 | 0.3622 |
| | 0.5 | 525 | 0.5700 | 13 | 1.0905 | 4 | 1.1485 | 2 | 0.9845 | 1 | 0.7800 |
| | 1 | 589 | 1.0450 | 22 | 1.5655 | 4 | 1.6217 | 2 | 1.4456 | 1 | 1.1979 |

Table 3

Optimum value of $L$ and corresponding value of $\lambda_2$

| $\mu_1$ | 2 | 4 | 6 | 8 | 10 | **12** | 14 | 16 |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{R}_f(\mu_1)$ | 6.4723 | 12.6282 | 22.8559 | 27.8689 | 29.3834 | **29.6287** | 29.4716 | 29.2023 |

Table 4

Effect of $\mu_1$ on $\mathcal{R}_f(\mu_1)$ for $(c, L, \lambda_1, \mu_2, p, \rho) = (4, 10, 2, 2.5, 0.7, 0.9)$