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MAP/PH/1 retrial queueing-inventory system with orbital search and reneging of customers

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Abstract. A single server retrial queueing-inventory is considered in which customers join directly to the orbit according to a Markovian arrival process (MAP). Service time of customers are independent and identical distributed phase-type distributed (PH) random variables. Inter retrial times are exponentially distributed with parameter $n\eta$ when n customers are in the orbit. Unsuccessful retrial customers tend to leave the system (impatience) with positive probability. In addition we also introduce search of orbital customers for next service with state dependent probability, immediately on current service completion. This system is shown to be always stable. We compute the long run system state probability.Under certain stringent conditions we prove that a particular case has a product form solution. We get explicit solutions to some retrial queueing models.

Keywords: queueing-inventory, stochastic decomposition, retrial, orbital search, reneging.

1. Introduction

In this paper we first discuss a general queueing-inventory model with correlated inter arrival time (Markovian arrival process (MAP)) and phase type distributed service time. Primary customers are directed to an orbit of infinite capacity from where they try to access the server. Impatience of such customers is taken into account. In addition we introduce search for orbital customers at the end of a service, provided there is at least one item left in inventory and the orbit is nonempty. We prove that the system is always stable. Next we proceed to a special case of the system described above, for which we produce a stochastic decomposition by constructing an appropriate blocking set.

Blocking sets (more aptly partial blocking sets) have been discussed in Krenzler and Daduna [5], among several other researchers, to produce product form solution (see for example Schwarz et al. [12], Saffari et al. [11], Krishnamoorthy and Viswanath [8], Baek and Moon [2]). A discussion on optimal blocking set could be found in Krishnamoorthy et al. [6]. These are in the classical queueing inventory context. Of these Schwarz et al. [12] was the first to obtain product form solution. Krishnamoorthy and Viswanath [8] extended it to the production inventory ser up and Saffari et al. [11] to the case of arbitrarily distributed lead time. For a survey of investigation on queueing - inventory process one may refer to Krishnamoorthy et al. [7].

Investigation on stochastic decomposition of retrial - queueing - inventory had not produced the desired result for the past one decade. This could be basically attributed to the fact that an appropriate blocking set was evading the researchers. One of the objectives of this paper is to produce such a blocking set and thereby achieve the desired result. In this process we also provide a geometric distribution, except for the multiplicative constants, for the 'modified' retrial queueing process that we introduced to produce stochastic decomposition.

In the present paper attached inventory is controlled by the (s, S) policy. Neverthless, we can extend the results obtained here to the control policies such as $(s, Q), (S-1, S)$ and random order cases. The (s, S) policy is as follows: Assume that each customer demands exactly one unit of the item at the end his service. From S, when $(S-s)$ items are supplied to customers, the inventory level reaches s which triggers an order placement for replenishment. As and when order materialization takes place (lead time is exponentially distributed with parameter β), the quantity replenished is that much to bring the inventory level to S.

The salient features of this paper are (i) search of orbital customers with positive probability immediately on completion of a service, provided at least one item in inventory is available and at least one customer is present in the orbit (the service system will be aware of this if a registry of orbital customers is available); the search time is assumed to be negligible, (ii) customers tend to renege on retrial if the server is found to be busy, (iii) retrial rate is linear, which indicates that the system behaves like an ordinary queue when number of customers in orbit is very large and (iv) the special case we considered in Section 4 leads us to product form solution in retrial queueing-inventory, though under very stringent conditions. In addition the effect of positive and negative correlations on the arrival process provides some interesting insight into the system.

The rest of this paper is arranged as follows. Mathematical modelling and analysis of the problem in the general setup is taken up in Section 2. Numerical illustrations of the performance measures are provided in Section 3. The effect of positive and negative correlation of the arrival process are extensively discussed. Section 4 is on a special case of the problem discussed in Section 2. Stochastic decomposition of this model is also provided there. Some special cases of retrial queueing models are also discussed.

Notations and abbreviation used in the sequel:

$\mathbf{e}$	column vector of 1's with appropriate order
$\mathbf{0}$	vector consisting of 0's with appropriate dimension
O	zero matrix with appropriate order
$N(t)$	number of customers in the system at time t
$I(t)$	number of items in the inventory at time t
$J_1(t)$	phase of service
$J_2(t)$	phase of arrival process
$C(t)$	status of server at time $t = \begin{cases} 0 & \text{server idle} \\ 1 & \text{server busy} \end{cases}$

2. Model Description

Consider a retrial queueing-inventory system with a single server to which customers arrive according to a Markovian arrival process (MAP) with representation (D_0, D_1) of order m (see Chakravarthy [3]). Let $\boldsymbol{\theta}$ be the steady-state probability vector of $D = D_0 + D_1$. Then $\boldsymbol{\theta} D = 0$ and $\boldsymbol{\theta}\mathbf{e} = 1$. The fundamental rate λ of this MAP is given by $\lambda = \boldsymbol{\theta} D_1 \mathbf{e}$ which gives the expected number of arrivals per unit of time. The coefficient of correlation c_{cor} of time intervals between successive arrivals is given as $c_{cor} = (\lambda\boldsymbol{\theta}(-D_0)^{-1}D_1(-D_0)^{-1}\mathbf{e} - 1)/c_{var}$. External arrivals enter directly to an orbit of infinite capacity as in Neuts and Rao [10]. The interval between two successive repeated attempts is exponentially distributed with parameter $n\eta$ when there are n customers in the orbit. After an unsuccessful retrial he rejoins the orbit with probability p or leaves the system without waiting for service with probability $q = 1 - p$. The service time is phase type distributed with representation (α, T) of order r such that $\mathbf{T}^0 = -T\mathbf{e}$ (see Neuts [9]). At the end of a service that customer is provided one item from the inventory. The (s, S)- control policy is adopted. The lead time for replenishment follows an exponential distribution with parameter β. Immediately after a service completion, the server goes in search of a customer from the orbit (see Artalejo et al. [1]) with probability p_n $(p_0 = 0)$ which depends on the number n of customers in the orbit, if at least one item is available in the inventory. With probability $q_n = 1 - p_n$ the server remains idle. The search time is assumed to be negligible. $\Omega' = \{(N(t), C(t), I(t), J_1(t), J_2(t)), t \geq 0\}$ is a continuous time Markov chain $(CTMC)$ which is a level dependent quasi-birth and death process $(LDQBD)$ with state space

$$\{(n, 0, i, k); n \geq 0, 0 \leq i \leq S, 1 \leq k \leq m\} \bigcup$$
$$\{(n, 1, i, j, k); n \geq 0, 1 \leq i \leq S, 1 \leq j \leq r, 1 \leq k \leq m\}.$$

The infinitesimal generator of Ω' is of the form

$$\mathcal{Q}' = \begin{pmatrix} \tilde{\mathcal{B}}_{1,0} & \tilde{\mathcal{B}}_0 & & & \\ \tilde{\mathcal{B}}_{2,1} & \tilde{\mathcal{B}}_{1,1} & \tilde{\mathcal{B}}_0 & & \\ & \tilde{\mathcal{B}}_{2,2} & \tilde{\mathcal{B}}_{1,2} & \tilde{\mathcal{B}}_0 & \\ & & \tilde{\mathcal{B}}_{2,3} & \tilde{\mathcal{B}}_{1,3} & \tilde{\mathcal{B}}_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$\tilde{\mathcal{B}}_{2,n} = \begin{pmatrix} \tilde{\mathcal{B}}_{2,n}^{(0,0)} & \tilde{\mathcal{B}}_{2,n}^{(0,1)} \\ O & \tilde{\mathcal{B}}_{2,n}^{(1,1)} \end{pmatrix}, \; n \geq 1,$$

$$\tilde{\mathcal{B}}_{1,n} = \begin{pmatrix} \tilde{\mathcal{B}}_{1,n}^{(0,0)} & O \\ \tilde{\mathcal{B}}_{1,n}^{(1,0)} & \tilde{\mathcal{B}}_{1,n}^{(1,1)} \end{pmatrix}, \; n \geq 0,$$

$$\tilde{\mathcal{B}}_0 = \begin{pmatrix} \tilde{\mathcal{B}}_0^{(0,0)} & O \\ O & \tilde{\mathcal{B}}_0^{(1,1)} \end{pmatrix}.$$

with

$$\left(\tilde{\mathcal{B}}_{2,n}^{(0,0)}\right)_{ij} = \begin{cases} nq\eta I_m & \text{for } j = i, \quad i = 1 \\ O & \text{otherwise} \end{cases}$$

$$\left(\tilde{\mathcal{B}}_{2,n}^{(0,1)}\right)_{ij} = \begin{cases} \alpha \otimes n\eta I_m & \text{for } j = i-1, \quad 2 \leq i \leq S+1 \\ O & \text{otherwise} \end{cases}$$

$$\left(\tilde{\mathcal{B}}_{2,n}^{(1,1)}\right)_{ij} = \begin{cases} nq\eta I_{rm} & \text{for } j = i, & 1 \leq i \leq S \\ \mathbf{T}^0 \otimes \alpha \otimes p_n I_m & \text{for } j = i-1, & 2 \leq i \leq S \\ O & \text{otherwise} \end{cases}$$

$$\left(\tilde{\mathcal{B}}_{1,n}^{(0,0)}\right)_{ij} = \begin{cases} D_0 - (nq\eta + \beta) I_m & \text{for } j = i, & 1 \leq i \leq s+1 \\ D_0 - nq\eta I_m & \text{for } j = i, & s+2 \leq i \leq S+1 \\ \beta I_m & \text{for } j = S+1, & 1 \leq i \leq s+1 \\ O & \text{otherwise} \end{cases}$$

$$\left(\tilde{\mathcal{B}}_{1,n}^{(1,0)}\right)_{ij} = \begin{cases} \mathbf{T}^0 \otimes I_m & \text{for } j = i, & i = 1 \\ \mathbf{T}^0 \otimes q_n I_m & \text{for } j = i, & 2 \leq i \leq S \\ O & \text{otherwise} \end{cases}$$

$$\left(\tilde{\mathcal{B}}_{1,n}^{(1,1)}\right)_{ij} = \begin{cases} (T \oplus D_0) - (\beta + nq\eta)I_{rm} & \text{for } j = i, \quad 1 \le i \le s \\ (T \oplus D_0) - nq\eta I_{rm} & \text{for } j = i, \quad s+1 \le i \le S \\ \beta I_{rm} & \text{for } j = S, \quad 1 \le i \le s \\ O & \text{otherwise} \end{cases}$$

$$\left(\tilde{\mathcal{B}}_{0}^{(0,0)}\right)_{ij} = \begin{cases} D_1 & \text{for } j = i, \quad 1 \le i \le S+1 \\ O & \text{otherwise} \end{cases}$$

$$\left(\tilde{\mathcal{B}}_{0}^{(1,1)}\right)_{ij} = \begin{cases} I_r \otimes D_1 & \text{for } j = i, \quad 1 \le i \le S \\ O & \text{otherwise} \end{cases}$$

We show that the above described system is always stable (see Tweedie [13]).

2.1. Stability condition

Theorem 2.1. *Let $\{X(t), t \ge 0\}$ be a Markov chain with state space $\mathcal{S}$ and rates of transition g_{xy}, $x, y \in \mathcal{S}$, $\sum_y g_{xy} = 0$. Assume that there exist*

1. *a function $\psi(x)$, $x \in \mathcal{S}$, which is bounded from below (Lyapunov or test function)*
2. *a positive number ϵ such that*
   (i) *Variable $w_x \le \sum_{y \ne x} g_{xy}(\psi(y) - \psi(x)) < \infty$ for all $x \in \mathcal{S}$.*
   (ii) *$w_x \le -\epsilon$ for all $x \in \mathcal{S}$ except perhaps a finite number of states.*

Then the Markov process $\{X(t), t \ge 0\}$ is regular and ergodic.

Proof. Construct an appropriate Lyapunov function and use it to show that the mean drift is negative. □

2.2. Steady state analysis

Since the model described in the previous section is an LDQBD, we use an algorithmic solution based on Neuts - Rao Truncation process (see Neuts and Rao [10]). Application of this method modifies the process Ω' into the process $\tilde{\Omega}'$ with infinitesimal generator $\tilde{\mathcal{Q}}'$ where

$$\tilde{\mathcal{B}}_{1,N} = \tilde{\mathcal{B}}_1 \text{ and } \tilde{\mathcal{B}}_{2,N} = \tilde{\mathcal{B}}_2 \text{ for } n \ge N \text{ in } \mathcal{Q}'.$$

system state probabilities of n and $n+1$ differ by less than ϵ for $n \ge N$.

Let ς be the steady state probability vector of $\tilde{\mathcal{Q}}'$. Then

$$\varsigma\tilde{\mathcal{Q}}' = \mathbf{0}, \varsigma\mathbf{e} = 1.$$

Writing $\varsigma_n = (\varsigma_n(0), \varsigma_n(1))$ where $\varsigma_n(0)$ and $\varsigma_n(1)$ are probability vectors corresponding to the server idle and busy status respectively, with n customers in the orbit.

From the repeating part of $\tilde{\mathcal{Q}}'$ we can write

$$\varsigma_{N+k-1} = \varsigma_{N-1}R^k, \ k \geq 1$$

where R is the minimal non-negative solution of the matrix quadratic equation

$$R^2\tilde{\mathcal{B}}_2 + R\tilde{\mathcal{B}}_1 + \tilde{\mathcal{B}}_0 = \mathbf{0}.$$

Again $\varsigma\tilde{\mathcal{Q}}' = \mathbf{0}$ leads to

$$\varsigma_n = \varsigma_{n-1}R_n, \ 1 \leq n \leq N-1$$

where

$$R_n = -\tilde{\mathcal{B}}_0\left[\tilde{\mathcal{B}}_{1,n} + R_{n+1}\tilde{\mathcal{B}}_{2,n+1}\right]^{-1}, \ 1 \leq n \leq N-2$$

and

$$R_{N-1} = -\tilde{\mathcal{B}}_0\left[\tilde{\mathcal{B}}_{1,N-1} + R\tilde{\mathcal{B}}_2\right]^{-1}.$$

Now from the normalizing condition we have

$$\varsigma_0\left[I + \sum_{i=1}^{N-2}\prod_{j=1}^{i}R_j + \prod_{j=1}^{N-1}R_j(I-R)^{-1}\right]\mathbf{e} = 1.$$

Next we proceed to compute some system state characteristics.

2.3. Performance measures

— Expected number of customers in the orbit $N_O = \sum_{n=1}^{\infty} n\varsigma_n\mathbf{e}$.

— Probability that the server is idle $P_{idle} = \sum_{n=0}^{\infty}\varsigma_n(0)\mathbf{e}$.

— Probability that the server is busy $P_{busy} = \sum_{n=0}^{\infty}\varsigma_n(1)\mathbf{e}$.

— Fraction of time inventory is empty $P_0 = \sum_{n=0}^{\infty}\varsigma_n(0,0)\mathbf{e}$.

— Expected number of customers reneging per unit time due to no item in the inventory

$$E_{loss} = \frac{\eta}{\beta} \sum_{n=1}^{\infty} n\varsigma_n(0,0)\mathbf{e}$$

— Expected rate of successful retrial $E_{SR} = \eta \sum_{n=1}^{\infty} \sum_{i=1}^{S} n\varsigma_n(0,i)\mathbf{e}$.

— Expected number of items in the inventory

$$E_I = \sum_{n=0}^{\infty} \sum_{i=1}^{S} i \left[\varsigma_n(0,i)\mathbf{e} + \varsigma_n(1,i)\mathbf{e}\right].$$

— Expected replenishment rate

$$E_{RR} = \beta \sum_{n=0}^{\infty} \left[\sum_{i=0}^{s} \varsigma_n(0,i)\mathbf{e} + \sum_{i=1}^{s} \varsigma_n(1,i)\mathbf{e}\right].$$

In the next section the above measures are numerically illustrated and useful conclusions are drawn.

3. Numerical illustration

For the arrival process, we consider two sets of distinct values for D_0 and D_1. The arrival processes labeled $MAP^{(N)}$ and $MAP^{(P)}$ respectively, have negative and positive correlations with values -0.4889 and 0.4889 and covariance 1.9867.

We fix parameters $r = 2, m = 3, \boldsymbol{\alpha} = (0.8 \;\; 0.2)$,

$$T = \begin{bmatrix} -12 & 9 \\ 8 & -16 \end{bmatrix}, \mathbf{T}^0 = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

1. MAP with negative correlation ($MAP^{(N)}$):

$$D_0 = \begin{pmatrix} -5.011 & 5.011 & 0 \\ 0 & -5.011 & 0 \\ 0 & 0 & -1128.75 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.0501 & 0 & 4.9609 \\ 1117.463 & 0 & 11.288 \end{pmatrix}$$

2. MAP with positive correlation ($MAP^{(P)}$):

$$D_0 = \begin{pmatrix} -5.011 & 5.011 & 0 \\ 0 & -5.011 & 0 \\ 0 & 0 & -1128.75 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 4.9609 & 0 & 0.0501 \\ 11.288 & 0 & 1117.463 \end{pmatrix}$$

Tables 1(a) and 1(b) respectively, give the effect of negative and positive correlated inter-arrival times on system characteristics for different values of probability q, of customer leaving the system on unsuccessful retrial, other parameters being fixed as indicated below the tables. It is interesting to observe that all measures have higher values for positively correlated MAP than that for the negatively correlated MAP.

The expected replenishment rate E_{RR}, for MAP with positive correlation is lower compared to MAP with negative correlation which may be attributed to higher inventory level in the former. However, when the orbital search probability p_n is independent of n (that is, $p_n \equiv \mathcal{P}$), and then as $\mathcal{P}$ is varied, the variations in the expected number of customers in the orbit is not that pronounced unlike that for varying q values. The server idle probability shows marked decrease (see Tables 2(a) and 2(b)).

q	N_O	P_{idle}	E_{SR}	E_I
1	0.4696	0.5155	0.8854	5.9147
0.9	0.4945	0.5058	0.8851	5.9046
0.8	0.5245	0.4944	0.8844	5.8928
0.7	0.5616	0.4812	0.8831	5.8788
0.6	0.6088	0.4654	0.8810	5.8619
0.5	0.6713	0.4463	0.8774	5.8412
0.4	0.7580	0.4226	0.8715	5.8148
0.3	0.8876	0.3921	0.8615	5.7800
0.2	1.1024	0.3516	0.8440	5.7314
0.1	1.5158	0.2975	0.8150	5.6607

(a)

Effect of q: $MAP^{(N)}$

q	N_O	P_{idle}	E_{SR}	E_I
1	3.2187	0.6091	1.4704	5.9663
0.9	3.5646	0.6043	1.4657	5.9602
0.8	3.9606	0.5987	1.4602	5.9532
0.7	4.4188	0.5922	1.4536	5.9451
0.6	4.9579	0.5845	1.4456	5.9355
0.5	5.6098	0.5752	1.4355	5.9240
0.4	6.4334	0.5635	1.4225	5.9096
0.3	7.5500	0.5483	1.4046	5.8907
0.2	9.2524	0.5271	1.3784	5.8639
0.1	12.4701	0.4936	1.3346	5.8205

(b)

Effect of q: $MAP^{(P)}$

Table 1

Effect of q: Fix $S = 8, s = 4, \eta = 9, \beta = 3, p_n = 0.75$ for $n \geq 1$

3.1. Effect of arrival process

For the arrival process, we consider the following four sets of values for D_0 and D_1. In the next experiment we take distinct MAPs - Erlang, Hyper-exponential, all of which have zero correlation in the inter-arrival time and then we have one positively correlated and another negatively correlated MAPs. The effect of these MAPs on the expected number of customers in orbit show moderate variation with respect to $\mathcal{P}(q)$, except the MAP with positive correlation and that too with increase in reneging probability (see Table 3).

$\mathcal{P}$	N_Q	P_{idle}	E_{SR}	E_I
1	0.9036	0.3721	0.5966	5.2943
0.9	0.9310	0.3762	0.7133	5.2962
0.8	0.9586	0.3804	0.8296	5.2981
0.7	0.9863	0.3845	0.9455	5.2999
0.6	1.0142	0.3887	1.0608	5.3017
0.5	1.0422	0.3930	1.1756	5.3034
0.4	1.0703	0.3972	1.2898	5.3051
0.3	1.0985	0.4015	1.4034	5.3068
0.2	1.1267	0.4057	1.5162	5.3084
0.1	1.1551	0.4100	1.6283	5.3100
0	1.1835	0.4143	1.7396	5.3116

(a)

Effect of $\mathcal{P}$: $MAP^{(N)}$

$\mathcal{P}$	N_Q	P_{idle}	E_{SR}	E_I
1	10.6601	0.5265	1.2614	5.4087
0.9	10.7482	0.5281	1.3142	5.4082
0.8	10.8362	0.5298	1.3676	5.4078
0.7	10.9240	0.5314	1.4218	5.4073
0.6	11.0116	0.5331	1.4766	5.4068
0.5	11.0991	0.5347	1.5321	5.4063
0.4	11.1865	0.5364	1.5883	5.4058
0.3	11.2738	0.5381	1.6453	5.4053
0.2	11.3610	0.5398	1.7031	5.4047
0.1	11.4482	0.5415	1.7617	5.4041
0	11.5353	0.5432	1.8211	5.4035

(b)

Effect of $\mathcal{P}$: $MAP^{(P)}$

Table 2

Effect of $p_n = \mathcal{P}$ for $n \geq 1$: Fix $S = 8, s = 3, \eta = 9, \beta = 3, q = 0.25$

1. Erlang (ERL):

$$D_0 = \begin{pmatrix} -10 & 10 \\ 0 & -10 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 \\ 10 & 0 \end{pmatrix}$$

2. Hyper-exponential (HYP):

$$D_0 = \begin{pmatrix} -9.5 & 0 \\ 0 & -0.95 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 8.55 & 0.95 \\ 0.855 & 0.095 \end{pmatrix}$$

3. MAP with negative correlation ($MAP^{(N)}$):

$$D_0 = \begin{pmatrix} -5.011 & 5.011 & 0 \\ 0 & -5.011 & 0 \\ 0 & 0 & -1128.75 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.0501 & 0 & 4.9609 \\ 1117.463 & 0 & 11.288 \end{pmatrix}$$

4. MAP with positive correlation ($MAP^{(P)}$):

$$D_0 = \begin{pmatrix} -5.011 & 5.011 & 0 \\ 0 & -5.011 & 0 \\ 0 & 0 & -1128.75 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 4.9609 & 0 & 0.0501 \\ 11.288 & 0 & 1117.463 \end{pmatrix}$$

The above MAP processes will be normalized so as to have a specific arrival rate. However, these are qualitatively different in that they have different variance and correlation structure. The first two arrival processes, namely, ERL and HYP have zero correlation for two successive inter-arrival times. The arrival processes labeled $MAP^{(N)}$ and $MAP^{(P)}$, respectively, have negative and positive correlation for two successive inter-arrival times with values -0.4889 and 0.4889. The covariance of these four arrival processes are, respectively, 0.5, 5.0388, 1.9867 and 1.9867.

Fix $(S, s, \eta, \beta) = (8, 4, 9, 3), r = 2, \boldsymbol{\alpha} = (0.8 \;\; 0.2), T = \begin{bmatrix} -12 & 9 \\ 8 & -16 \end{bmatrix}$, $\mathbf{T}^0 = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$.

	$\mathcal{P}$				q			
	ERL	HYP	$MAP^{(N)}$	$MAP^{(P)}$	ERL	HYP	$MAP^{(N)}$	$MAP^{(P)}$
0.1	1.0133	1.1071	1.0624	7.9159	1.4319	1.5748	1.5158	12.4701
0.2	0.9833	1.0844	1.0351	7.8598	1.0227	1.1825	1.1024	9.2524
0.3	0.9536	1.0615	1.008	7.8038	0.8254	0.957	0.8876	7.55
0.4	0.9244	1.0385	0.9809	7.7477	0.7115	0.8094	0.758	6.4334
0.5	0.8955	1.0153	0.9541	7.6916	0.637	0.7066	0.6713	5.6098
0.6	0.8671	0.9921	0.9274	7.6355	0.5842	0.6316	0.6088	4.9579
0.7	0.8392	0.9687	0.9008	7.5793	0.5446	0.5749	0.5616	4.4188
0.8	0.8116	0.9453	0.8745	7.5231	0.5137	0.5307	0.5245	3.9606
0.9	0.7845	0.9218	0.8483	7.4668	0.4888	0.4953	0.4945	3.5646
1	0.7579	0.8982	0.8223	7.4105	0.4684	0.4663	0.4696	3.2187

Table 3

Effect of arrival process on N_O

3.2. Cost Analysis

Now for imposing a cost associated with the system under study, we introduce a cost function $\mathcal{F}(q, \mathcal{P})$ defined by

$$\mathcal{F}(q, \mathcal{P}) = C_1 E_{RR} + C_2 N_O + C_3 E_I + C_4 P_0 - C_5 E_{SR}$$

where
C_1 =Cost of inventory procurement per item
C_2 =Cost of holding customers for one unit of time
C_3 =Cost of holding inventory for one unit of time
C_4 =Cost per unit time due to an idle server
C_5 =Revenue per unit time due to successful retrial

In order to study the cost function we first fix $r = 2, \boldsymbol{\alpha} = (0.8 \;\; 0.2)$, $(C_1, C_2, C_3, C_4, C_5, S, s, \eta, \beta) = (\$50, \$0.25, \$0.5, \$5, \$15, 8, 3, 9, 3)$,

$$T = \begin{bmatrix} -12 & 9 \\ 8 & -16 \end{bmatrix}, \mathbf{T}^0 = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

$\mathcal{P}$	$MAP^{(N)}$	$MAP^{(P)}$
0	4.9149	3.8687
0.1	6.7179	4.8082
0.2	8.5366	5.7334
0.3	10.3703	6.6446
0.4	12.2180	7.5424
0.5	14.0789	8.4270
0.6	15.9520	9.2990
0.7	17.8364	10.1586
0.8	19.7311	11.0062
0.9	21.6353	11.8423
1	23.5481	12.6672

(a)

Effect of $\mathcal{P}$: fix $q = 0.2$

q	$MAP^{(N)}$	$MAP^{(P)}$
1	13.4946	4.0310
0.9	13.8000	4.4113
0.8	14.1551	4.8463
0.7	14.5736	5.3515
0.6	15.0750	5.9498
0.5	15.6885	6.6777
0.4	16.4586	7.5962
0.3	17.4556	8.8179
0.2	18.7825	10.5839
0.1	20.5201	13.5842

(b)

Effect of q: fix $\mathcal{P} = 0.75$

Table 4

Cost function

This cost is evaluated and compared in Table 4(a) and 4(b) between MAP with positive and negative correlations for distinct $\mathcal{P}$ (Table 4(a)) and q (Table 4(b)) values. Service time is taken to be phase type distributed with representation $(\boldsymbol{\alpha}, T)$ as indicated above. The effect on (cost) of $\mathcal{P}$ and q are seen to be markedly pronounced for MAP with negative correlation than that with positive correlation. This could be attributed to the higher share of replenishment cost in the former than that in the latter.

4. Special Case

Next we analyze a special case of the system discussed in sections 2 and 3. This special case provides product form solution to the system state. The basic difficulty in arriving at product form solution is identifying a suitable blocking set. We produce here such a set; the assumptions leading to that set turn out to be too strong. On establishing stochastic decomposition we pass on to certain special class of retrial queues for which closed form solution is derived(both constant and classical retrial cases are considered).

The MAP in Section 2 is now replaced by a Poisson process of rate λ. Service time is exponentially distributed with parameter μ. For the purpose of producing a stochastic decomposition of the system state, we restrict the arrival of customers as follows: All primary arrivals (external customers) must join an orbit of infinite capacity on arrival, from where, through retrial alone, they can access the server (as in Neuts and Rao [10]). Further if the server is busy at the time an external arrival takes place, then that external customer does not join the system. The customer at the head of the orbit alone accesses the server through retrial (see Gomez-Corral [4]). The interval between two successive repeated attempts is exponentially distributed with parameter η. Another crucial assumption that we make, as in Schwarz et al. [12], Saffari et al. [11], Krishnamoorthy and Viswanath [8], is that when inventory level is zero, no primary customer joins orbit or orbital customers retry (even if they do so, they return to orbit). As an example for the model under study we may think of a polling model where the gate is closed the moment the server starts service at a node. The search of orbital customers is done away in this special case. Then $\Omega = \{(N(t), C(t), I(t)), t \geq 0\}$ forms a $CTMC$ with state space $\{(n, 0, i); n \geq 0, 0 \leq i \leq S\} \bigcup \{(n, 1, i); n \geq 0, 1 \leq i \leq S\}$ which is an $LIQBD$ process.

The infinitesimal generator is of the form

$$\mathcal{Q} = \begin{bmatrix} A_{00} & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

such that $\mathbf{x}\mathcal{Q} = 0$ and $\mathbf{xe} = 1$. Each matrix A_{00}, A_0, A_1, A_2 is a square matrix of order $(2S + 1)$. The entries of the block matrix are clear from the context.

The Markov chain is stable if and only if (see Neuts [9]) the left drift rate exceeds the right drift rate. That is,

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}.$$

Thus we have the following lemma.

Lemma 4.1. *The system under study is stable if and only if $\lambda < \eta$.*

Theorem 4.2. *Under the necessary and sufficient condition $\lambda < \eta$, for stability, we get*

$$x_n(k,i) = \begin{cases} \left(1-\frac{\lambda}{\eta}\right)\left(\frac{\lambda}{\eta}\right)^n \xi(k,i), & k=0, 0 \le i \le S, \\ \left(1-\frac{\lambda}{\eta}\right)\left(\frac{\lambda}{\eta}\right)^n \xi(k,i), & k=1, 1 \le i \le S \end{cases}$$

where

$$\xi(k,i) = \begin{cases} \frac{\beta}{\lambda}(a)^i (b)^{i-1} \xi(0,0), & k=0, 1 \le i \le s, \\ \frac{\beta}{\lambda}(a)^s (b)^s \xi(0,0), & k=0, s+1 \le i \le S-1, \\ \left(\frac{\beta}{\lambda}(a)^s (b)^s + \frac{\beta}{\beta+\lambda+\mu}\left[1-(a)^s (b)^s\right]\right)\xi(0,0), & k=0, i=S, \\ \frac{\beta}{\mu}(a)^{i-1} (b)^{i-1} \xi(0,0), & k=1, 1 \le i \le s, \\ \frac{\beta}{\mu}(a)^s (b)^s \xi(0,0), & k=1, s+1 \le i \le S \end{cases}$$

with

$$\xi(0,0) = \left\{(a)^s (b)^s (\lambda+\mu)\left[\frac{1}{\beta+\lambda+\mu} + (S-s)\frac{\beta}{\lambda\mu}\right] + \frac{\beta}{\beta+\lambda+\mu}\right\}^{-1}$$

where $a = \frac{\beta+\mu}{\mu}$ *and* $b = \frac{\beta+\lambda}{\lambda}$.

Now we consider a pure queueing situation arising out of the retrial queueing-inventory model. In other words suppose that the material for service is abundantly available. In this case we ignore the inventory status and examine only the server status and the number of customers in the orbit. As in Neuts and Rao [10], we assume that primary customers do not access the server directly; instead they first join an orbit of infinite capacity, from where they access the server according to the FIFO discipline. Primary customers do not join the orbit when server is busy. On retrial by the head of orbital queue, if the server is found busy, the customer returns to orbit. With these assumptions we get the following important corollary for the retrial queue under consideration.

Corollary 4.3. *For the retrial queue under consideration we deduce the following system state distribution:*
Probability that i customers are in the orbit when the server is busy is $\left(1-\frac{\lambda}{\eta}\right)\left(\frac{\lambda}{\eta}\right)^i \left(\frac{\lambda}{\lambda+\mu}\right)$ *for* $i \ge 0$.
Probability that i customers are in the orbit but the server is idle is $\left(1-\frac{\lambda}{\eta}\right)\left(\frac{\lambda}{\eta}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)$ *for* $i \ge 0$.

We extend the above corollary (Corollary 4.3) to the case when retrial rate is linear. This means that the retrial rate is $n\eta$ when n customers are in the orbit (it may be FIFO discipline or all customers trying to access the server). Then we have the following result.

Theorem 4.4. *In the case of linear retrial rate of orbital customers, the long run system state probability is given by*

$$\frac{1}{n!}\left(\frac{\lambda}{\eta}\right)^n e^{-\lambda/\eta}\left(1+\frac{\lambda}{\mu}\right)^{-1} \text{ for } n \text{ customers in the orbit with server idle}$$

and

$$\frac{1}{n!}\frac{\lambda}{\mu}\left(\frac{\lambda}{\eta}\right)^n e^{-\lambda/\eta}\left(1+\frac{\lambda}{\mu}\right)^{-1} \text{ for } n \text{ customers in the orbit with server busy.}$$

We define the system reliability as the probability of a customer joining the system. A customer from outside joins the orbit only when the server is idle and at least one item is available in the inventory. The objective of a system designer is to maximize the probability: $P_{join} = \sum_{n=0}^{\infty}\sum_{j=1}^{S} x_n(0,j) = P_0 - \sum_{n=0}^{\infty} x_n(0,0)$, P_0 being the probability of an idle server.

5. Conclusions

In this paper a $MAP/PH/1$ queueing-inventory problem with exponentially distributed lead time was considered. We introduced impatience of orbital customers, which in turn resulted in the system being stable always. Search for orbital customers, immediately on completion of a service resulted in shorter expected waiting time than that in models without this search.

In a special case we produced explicit product form solution for a retrial - queueing - inventory problem. For the analysis (s, S) inventory control policy was followed. Other control policies also can be shown to yield the product form solution. Because of the highly non linear nature of the cost function in s and S, it is extremely difficult to prove that it is convex in both variables. However, our computational experience indicated that the cost function constructed is convex. For the arbitrarily distributed lead time case (see Saffari et al. [11]) also this special case should yield product form solution.

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