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Adapted statistical experiments with random change of time

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Abstract. We study statistical experiments with random change of time, which transforms a discrete stochastic basis in a continuous one. Adapted stochastic experiments are studied in continuous stochastic basis in the series scheme. The transition to limit by the series parameter generates an approximation of adapted statistical experiments by a diffusion process with evolution.

The average intensity parameter of renewal times are estimated in three different cases: the Poisson renewal process, stationary renewal process with delay and the general renewal process with Weibull-Gnedenko renewal time distribution.

Keywords: statistical experiment, random change of time, special semimartingale, triplet of characteristics, limit theorem, statistic parameter estimation.

Adapted statistical experiments with random change of time are realized by a counting renewal process $\nu(t)$, $t \geq 0$, with the Markov renewal moments:

$$\tau_k := \inf\{t : \nu(t) \geq k\} , \quad k \geq 0.$$

The renewal intervals $\theta_{k+1} := \tau_{k+1} - \tau_k$, $k \geq 0$, $\tau_0 = 0$ are determined by the distribution function $\Phi(t) = \mathsf{P}\{\theta_{k+1} < t\}$.

The normalized counting renewal process $\nu_N(t)$, $t \geq 0$, in the series scheme with $N \to \infty$, is defined by the time stretching: $\nu_N(t) := \nu(tN)$, $t \geq 0$.

The random change of time in a discrete stochastic basis

$$\mathfrak{B}_{\mathbb{N}} = (\Omega, \mathfrak{F}, (\mathfrak{F}_k, k \in \mathbb{N}), \mathcal{P})$$

is given by the filtration $\mathfrak{G}_t^N = \mathfrak{F}_{\nu_N(t)}$, $t \geq 0$.

The adapted statistical experiments with random change of time is determined by a solution of the following difference stochastic equation:

$$\Delta\alpha_N(\tau_{k+1}^N) = -V_0(\alpha_N(\tau_k^N))/N + \Delta\mu_N(\tau_{k+1}^N)/\sqrt{N} , \quad k \geq 0,$$

with the normalized regression function of increments:

$$V_0(c) = V(1 - c^2)(c - \rho) , \quad |c| \leq 1 , \quad |\rho| < 1 , \quad V > 0.$$

Using the notations

$$\alpha_k^N := \alpha_N(\tau_k^N) \ , \quad \mu_k^N := \mu_N(\tau_k^N),$$

the adapted statistical experiment $\alpha_N(t)$, $t \geq 0$, is characterized, as a special semimartingale, by three predictable characteristics [1, ch.2]:

— evolutionary component

$$V_t^N = -\frac{1}{N} \sum_{k=0}^{\nu_N(t)-1} V_0(\alpha_k^N) \ , \quad t \geq 0; \tag{1}$$

— variation of the stochastic component

$$\sigma_t^N = \frac{1}{\sqrt{N}} \sum_{k=0}^{\nu_N(t)} \sigma^2(\alpha_k^N)) \ , \quad \sigma^2(c) := 1 - V^2(c) \ , \quad t \geq 0,$$

— compensating measure of jumps

$$\Gamma_t^N(g) := \sum_{k=0}^{\nu_N(t)-1} \mathsf{E}[g(\Delta\alpha_{k+1}^N) \,|\, \mathfrak{F}_k^N] \ , \quad t \geq 0. \tag{2}$$

The limit theorem for the adapted statistical experiments is based on the canonical representation for semimartingales by the triplet of predictable characteristics (5) – (7). It is implemented in two stages.

Stage 1. The compactness of adapted statistical experiments (2) by $N \to \infty$ is established by using the approach [4] (see also [5]).

Stage 2. By additional conditions for predictable characteristics: the functions $V_0(c)$, $\sigma^2(c)$, $|c| \leq 1$ identify the limiting process, defined by the the limit predictable characteristics.

Theorem 1. *The adapted statistical experimens $\alpha_N(t)$, $t \geq 0$ in series scheme with the series parameter $N \to \infty$, determined by the predictable characteristics* (1) - (2) *with additional condition of convergence of initial values:*

$$\alpha_N(0) \xrightarrow{D} \alpha_0 \ , \quad \mathsf{E}\alpha_N(0) \to \mathsf{E}\alpha_0 \ , \quad N \to \infty.$$

converges, in distribution, to the diffusion process with evolution with scale change of time

$$\alpha_N(t) \xrightarrow{D} \alpha_0(t) \ , \quad 0 \leq t \leq T \ , \quad N \to \infty.$$

The predictable characteristics of the limiting process $\alpha^0(t)$, $t \geq 0$ has the following representation:

$$V_t^0 = \int_0^{qt} V_0(\alpha^0(u))du \ , \quad \sigma_t^0 = \int_0^{qt} \sigma^2(\alpha^0(u))du \ , \quad 0 \leq t \leq T.$$

and the compensating measure of jumps is absent:

$$\Gamma_t^N(g) \to 0 \ , \quad N \to \infty \ , \quad g(c) \in C_3(\mathbb{R}).$$

The limit diffusion process with evolution $\alpha^0(t) = \alpha_0(qt)$, $t \geq 0$, is given by the stochastic differential equation

$$d\alpha(t) = -V_0(\alpha(t))dt + \sigma(\alpha(t))dW_t \ , \quad t \geq 0, \tag{3}$$

with the linear time scaling

$$\alpha_0(t) = \alpha(qt) \ , \quad t \geq 0. \tag{4}$$

The scaling parameter is determined by the renewal intensity $q = 1/\mathsf{E}\theta_{k+1}$, $k \geq 0$.

The main statistical problem for the adapted statistical experiments is to estimate the scaling parameter q using the theory of renewal processes [6–8]

The simplest Poisson renewal process, as well as the stationary renewal process with delay, are characterized by the equality:

$$\mathsf{E}\nu(T) = qT \ , \quad T > 0.$$

At the same time, for a process with arbitrarily distributed renewal intervals

$$\Phi(t) = \mathsf{P}\{\theta_{k+1} \leq t\} \ , \quad k \geq 0,$$

the parameter q has estimation by using the strong law of large numbers [10, Ch.IV, §3]

$$\widehat{q} \approx \nu(T)/T.$$

The mentioned above statistical estimations has been numerically verified on the simulated trajectories of the renewal processes with previously fixed parameters in the cases discussed above.

Case 1: Poisson renewal process with parameter $q = 2$. In this case the nodal formula $\mathsf{E}\nu(t) = qt$ and its statistical interpretation $\mathsf{E}\widehat{\nu}(t) = \widehat{q}t$ implies the statistical estimates $\widehat{q}_T = \frac{1}{M}\sum_{m=1}^{M} \nu_m(T)/T$. Using exponential renewal interval generation $\theta_i = -(1/q)\ln(1-x_i)$, $x_i \in U(0,1)$, one stimulates the random samples of the level T hitting times of for the renewal process $\nu(T)$, in order to obtain the statistical estimates $\widehat{q}_T$:

Table 1

Estimate $\widehat{q_T}$ by $M = 10$

T=5	T=10	T=20	T=40	T=70	T=100	T=150	T=200
1,88	2,18	2,07	1,985	2,0057	2,028	2,018	1,9685

Case 2: The stationary renewal process with delay, determined by the initial distribution function of limit overjumps. Here a particular case is considered, with the initial renewal intervals calculated by the overjump a level T for enough big T. The other renewal intervals have Weibull-Gnedenko distribution function $W(1/2, 1/2)$: $\Phi(t) = 1 - e^{-\sqrt{2t}}$, $t \geq 0$. The strong law of large numbers provides the statistical estimation $\widehat{q} \approx \nu(T)/T$.

Considering that $1/q = \mathsf{E}\theta_k = (1/2)\Gamma(3) = 1$, hence $q = 1$.

The numerical simulation of the level T hitting times $\nu_m(T)$, $m = 1, 2, ...$ for the renewal process τ_k, $k \geq 1$, gives the following statistical estimates $\widehat{q}_T$:

Table 2

Estimate $\widehat{q_T}$

	$m = 1$	$m = 2$	$m = 3$
$T = 200$	1	0,935	1,02
$T = 300$	0,996666667	0,93	1,073333333
$T = 400$	1,075	0,945	1,0375

Case 3: Renewal processes with arbitrarily distributed renewal intervals. Here a particular case is considered, with Weibull-Gnedenko distribution function $W(2, 2)$: $\Phi(t) = 1 - e^{-t^2/4}$, $t \geq 0$. The strong law of large numbers provides the statistical estimation $\widehat{q} \approx \nu(T)/T$. Considering that $1/q = \mathsf{E}\theta_k = \lambda\Gamma(1, 5) = \sqrt{\pi}$, one has $q = 1/\sqrt{\pi} \approx 0,56419$.

Using W(2,2) renewal interval generation $\theta_k = 2 \cdot (-\ln(x_k))^{1/2}$, $x_k \in U(0, 1)$, one obtains the level T hitting times $\nu_m(T)$, $m = 1, 2, ...$ for the renewal process τ_k, $k \geq 1$, which gives the following statistical estimates $\widehat{q}_T$:

Table 3

Estimate $\widehat{q_T}$

	$m = 1$	$m = 2$	$m = 3$
$T = 300$	0,563333333	0,56	0,57
$T = 450$	0,566666667	0,557777778	0,566666667
$T = 580$	0,574137931	0,565517241	0,565517241
$T = 725$	0,569655172	0,55862069	0,56137931

Conclusion

The convergence of the adapted statistical experiments with random change of time to a limit diffusion process with evolution $\alpha_0(t)$, $t \geq 0$, given by the stochastic differential equation (3) with the linear time scaling (4), reduces the problem of the random change of time to a statistical estimation of only the average renewal intensity parameter q.

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