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Subjective Probability: Its Axioms and Operationalization

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Abstract. What sense can one make of the claim that "the probability of a nuclear accident is 10^{-5}"? Not surprisingly, the answer is difficult because there are many interpretations of quantified uncertainty, each burdened by its own baggage. This expository talk is a historical silk road style journey, which traces the development of quantified uncertainty from the days of Cardano to those of Kolmogorov, with stops at Bayes, Laplace, Bernoulli, de Moivre, Keynes, Venn, Borel, Ramsey, de Finetti, von Mises, von Neumann, Popper, and Savage.

An interpretation that is immune to a attack is that of Subjective Probability – to the philosopher Richard Jeffrey "The Real Thing". Here, probability is a corporate state of mind grounded in ones actions rather than an innate property of the physical world. This viewpoint, now at the very doorstep of *Quantum Theory*, is defended by notions of coherence and rationality, an elaborate system of preferences, consequences, and acts, to the claim that pure probability does not exist because it cannot be isolated from preferences. Hidden therein is the axiom of acrobatics (my term) which endeavours to operationalize subjective probability and in so doing levels the playing field.

The aim of this talk is to present a conversational overview of this topic.

Keywords: Chance, Classical Probability, Coherence, Exchangeability, Frequentist, Intuitive Probability, Logical Probability, Necessarist, Propensity.

1. Background: Probability and Statistics

The Pascal-Fermat correspondence of 1654 on problems of gambling stimulated further research on the topic by Bernoulli, who introduced the term *probabilitas* in the context of developing a new branch of science which he called *stochastics*, which in Greek and Latin stands for the science of prediction. To Bernoulli a relevant feature of stochastics is an event's readiness to occur, and the probability of an event is the degree of certainty of its occurrence. Thus Bernoulli's treatise *Ars Conjectandi*, or stochastics, is the art of measuring probabilities.

Bernoulli recognized the difficulty of determining the true value of probability, and this motivated him to develop his law of large numbers as an empirical method to determine an upper and lower limit of an unknown probability. de Moivre's *Doctrine of Chances* (1718) pertained to methods for calculating the probability of events using the addition multiplication and independence rules, without addressing the matter as to what probability is. Kolmogorov's aim in laying out an axiomatic foundation for probability was to establish a new branch of mathematics wherein a theory of probability pertains to a system of sets which satisfy certain conditions. He used the term probability detached from any real world meaning.

Simultaneous to Bernoulli's work, data sets were compiled by Graunt (1662) on demography, by Petty (1690) on political arithmetic, and by de Witt (1671) on actuarial mathematics. But most noteworthy is a 21 volume work by John Sinclair (between 1791 and 1799) entitled *Statistical Account of Scotland* wherein the term "statistics" appeared for the first time. Sinclair's motivation for using the German term "statistics" was to attract public attention by replacing Petty's "Political Arithmetic". The purpose of Sinclair's 21 volume work was to assess the political strength of a country via its inhabitant's happiness. Thus statistics is an artificial word which now stands for anything dealing with data.

2. Interpretations of Probability

The words probability and its synonyms like "chance" and "likelihood" have acquired several meanings, not clearly distinguishable from each other. But how can this explain how probability can be used in science, engineering, medicine, and economics where probability should have a clear and definite meaning in the context of its use? What is the connection between this abstract mathematical entity and the contexts of the above disciplines?

Three types of connectives have appeared: ***frequentist, necessarist***, and ***subjective***. To philosophers Carnap and Nagel probability admits both a frequentist and a necessarist interpretation, whereas to the mathematician Koopman both a necessarist and a subjectivist interpretation. In the frequentist interpretation probability is a unique, reproducible and physical property of a collective, and is the limit of a relative frequency. The necessarist view denies that probability statements are empirical. Here probability is a unique logical relationship between a proposition and a body of knowledge. In the subjectivist view probability is also a relation between a proposition and evidence but the relationship need not be purely logical nor unique. It is a quasi-logical relationship wherein probability is ones degree of belief about a proposition.

The subjectivist view due to Ramsey (1926) and de Finetti (1928) allows only certain combinations of degrees of belief in statements, the combinations governed by the well known Kolmogorov axioms of convexity, addition, and multiplication. Here a person can have any degree of belief based on his/her beliefs, and these beliefs can change with time even if the evidence remains the same. Thus probability of an event say $(X = x)$ is indexed by both time $\mathscr{T}$, and evidence $\mathscr{H}$ and is written $P^{\mathscr{T}}(X_1 = x_1; \mathscr{H})$. An individual's personal probability is revealed by his (her) disposition to bets (de Finetti) or actions under uncertainty (Ramsey).

Entering in the above mix are two other notions, ***intuitive probability*** and ***propensity***. The intuitive thesis due to Koopman (1940) holds that probability derives directly from intuition and is prior to objective experience, so that experience (or data) is to be interpreted in terms of probability, and not the reverse. The propensity notion of probability was

proposed by the American philosopher Pierce [cf. Singpurwalla (2016)]., and was developed by Popper (1957) in his attempts at providing an interpretation of quantum theory, different from the Heisenberg-Bohr subjectivist view. Here probability is a disposition or tendency of a physical situation to yield a long run relative frequency of an item. To Popper, propensities or ***chances*** are unobservable dispositional properties of the physical world, and depend on the generating conditions of an item. Thus the notion of propensity can be invoked for singular events, as well as for a collective, and are comparable to a Newtonian force. Popper's view about propensities align with Kolmogorov's view on probability as a primitive which depends on the experimental environment. Thus in the context of reliability, Kolmogorov (1969), p. 239 states: "..., the statistical law expressed by the (survival) curves is only a reflection of the law of probability connecting the useful life of a lamp with the materials and the conditions of manufacture". Furthermore, in discussing the law of large numbers about the closeness of x/n (the frequency) to p (the probability) Kolmogorov also says "will never allow us to be free of the necessity of referring to probabilities in the primitive imprecise sense of the term. By the word "primitive" it is here meant something that is understood without definition.

2.1. Criticisms of the Frequentist and Necessarist Theories

Of the several criticisms of the frequentist theory such as vagueness of terms in its definition, like "approximately", "similar conditions", and an "infinite number of trials", the most damaging one pertains to the fact that probability is a property of a collective and thus cannot be invoked on singular events, like the failure of a nuclear reactor. Here again the following quote from Kolmogorov (1963) is striking:

"The frequency concept based on the notion of limiting frequency as the number of trials increases to infinity does not contribute anything to substantiate the applicability of the results of probability theory to real practical problems where the number of trials is always finite. The frequency concept applied to a large but finite number of trials does not admit a rigorous exposition within the framework of pure mathematics".

The frequentist theory is therefore limited in scope, it main virtue being that it affords a simple example of quantities satisfying the laws of probability.

Classical probability, the version of probability most used by the founding fathers Beroulli, Bayes, de Moivre, and in almost all introductory texts is an archetypal example of the necessarist interpretation of probability, and is driven by the *principle of sufficient reason*. We say the probability of a coin landing heads in $1/2$ because there are only two possibilities, heads and tails, and they are equally likely, and thus should be assigned the same probability. Difficulties with this approach are:

i) the definition is circular because the term "likely" is used to define probability, and the term "probability" used to define likely.

ii) What if the coin is unbalanced?
iii) It is physically impossible to produce balanced coins, so the notion of balance is a subjective judgment anyway.

3. Subjective Probability, Utility, and Their Entanglement

Criticisms of the kind discussed in Section 2.1 gave birth to the notion of subjective probability by the likes of de Finetti, who, influenced by March, Einstein, and the theory of relativity which emphasized that physical quantities must be defined from the perspective of a observer using a given measuring instrument, wanted to do the same for quantifying uncertainty. Thus to de Finetti (1937), "the degree of probability exists only subjectively in the minds of individuals, and the probability attributed by an individual to a given event is revealed by the conditions under which he would be disposed to bet on the event". Similarly, Ramsey (1931) p. 170, who in the process of developing a logic for partial belief, spawned the notion of subjective probability, states "we are therefore driven to the supposition that the degree of a belief ... which we can express (vaguely) as the extent to which we are prepared to act on it". de Finetti in the 1920's operationalized subjective probability assuming a linear utility and no arbitrage via a 2-sided bet. Specifically, if an analyst $\mathscr{A}$ declares p as the probability of rain tomorrow to his(her) boss $\mathscr{L}$, then $\mathscr{A}$ is prepared to stake p units of money in exchange of one unit if it rains tomorrow, and is simultaneously also prepared to stake (1-p) units of money in exchange of one it if fails to rain. However it is $\mathscr{L}$ who gets to choose the side of this two-sided bet. Similarly, Ramsey (1926) took utility for granted, and sketched a proof for the existence of a choice-based subjective probability assuming that individuals make choices that maximize their utilities. Later on, in 1931, recognizing that the marginal utility of money diminishes, and also the aversion of many to betting, Ramsey puts forward his brilliant and celebrated proposal for a simultaneous axiomatization of a choice-based probability and utility based on a transitive system of preferences among choices under uncertainty.

Assuming the existence of a frequency based probability, von Neumann and Morgenstern in 1947 proved the existence of state-independent utility, and gave an axiomatic characterization of expected utility maximizers. This work was foundational in the sense that it enabled Savage (1954) to synthesize it together with that of de Finetti and Ramsey to introduce a new framework, and necessary and sufficient conditions for the simultaneous existence of a state independent utility and a finitely additive subjective probability as well as the characterization of expected utility maximizers. Also see Shafer (1986) for a constructive interpretation of subjective probability.

A consequence of the de Finetti-Ramsey-Savage work on subjective probability is the entanglement of the degree of pure belief from value. Thus, just like how it is impossible to measure position and momentum

with physical instruments, "belief and value are as inseparable as space and time" per Herman Rubin (1987). Pure probability uncontaminated by value is therefore a mental construct that makes sense only when it is regarded as a primitive (i.e. something that is understood without definition, and revealed by verbal reports). Thus in de Finetti's two volume book "***Probability Does Not Exist***".

4. Linking Subjective Probability and Propensity (Chance)

The word "chance" appears in two metaphysical senses: Popper's propensity (or pure probability) and von Mises' limit of a relative frequency. Both de Finetti and Ramsey have acknowledged chance as a useful entity, and the former by his *theorem on exchangeability* links it with subjective probability. Specifically, for an infinite binary exchangeable sequence $X_1, X_2, \cdots$,

$$P(X_1 = x_1, \cdots, X_n = x_n) = \int_0^1 p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}\Pi(p)dp,$$

where $p = \lim_{n\to\infty} \frac{\Sigma x_i}{n}$ exists, and
$\Pi(p)$ is a subjective (prior) probability density on an (objective) chance p.

This theorem serves as a foundation for Bayesian statistical inference, for without the assumption of exchangeability (or some version of it) inductive inference and prediction are not possible.

5. What is the Theory of Subjective Probability?

It is a theory which attempts to make precise the connection between ones coherent dispositions towards uncertainty and mathematical probability as axiomatized by Kolmogorov (1933). It accommodates the Bayes-Laplace classical interpretation, the intuitive views of Koopmans and Keynes, and the decision oriented approach of Ramsey, de Finetti, and Savage. Here each person needs to come up with numerical probabilities to sub-sets of a sample space to refect his(her) beliefs. Like Euclidian geometry, subjective probability begins with a set of axioms derived only after an examination of our intuitive notions of the subject. These axioms are basic qualitative relationships representing a person's judgments about preferences between acts or about ranking as to which events are more likely to occur than others. Subjective probabilities are a consequence of exercising the *principle of expected utility* invoked within several axiom systems.

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