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Factorization method in boundary crossing problems for random walks

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Abstract. We demonstrate an analytical approach to many problems related to crossing linear boundaries by random walk trajectories. Using factorization identities is the main instrument of the method.

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Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, $S_n = X_1 + \cdots + X_n$, $n \geq 1$. The sequence $\{S_n\}$ is usually called a random walk. Boundary crossing problems involve the study of distributions associated with reaching (or not reaching) the boundary of certain set for random walk trajectories.

Given a Borel set $B \subset \mathbb{R}$, introduce the first hitting time

$$N = \min\{n \geq 1 : \ S_n \in B\}.$$

Put $N = \infty$ if $S_n \in \overline{B} = \mathbb{R} \setminus B$ for all n.

We are interesting in the joint distribution of the pair (N, S_N) in the cases: $B = [b, \infty)$, $B = (-\infty, a]$ (one-sided problems, $a < 0$, $b > 0$), $B = (-\infty, a] \cup [b, \infty)$ (two-sided problem). The distribution of the sojourn time above a level and the distribution of the number of crossings of a strip by sample paths of a random walk are of our interest as well.

Introduce the double Laplace–Stieltjes transform (LST)

$$Q(z, \lambda) = \mathbf{E}\big(z^N e^{\lambda S_N}; \ N < \infty\big) = \sum_{n=1}^{\infty} z^n \int_B e^{\lambda y} \mathbf{P}(N = n, \, S_N \in dy),$$

and, in addition, the functions

$$Q_0(z, \lambda) = \sum_{n=1}^{\infty} z^n \mathbf{E}\big(e^{\lambda S_n}; \ N > n\big), \quad \varphi(\lambda) = \mathbf{E} e^{\lambda X}.$$

The following assertion (the main identity) is available [1, Ch. 18].

Theorem 1 *For $|z| < 1$ and $\operatorname{Re} \lambda = 0$ the following identity holds:*

$$\big(1 - z\varphi(\lambda)\big)\big(1 + Q_0(z, \lambda)\big) = 1 - Q(z, \lambda). \tag{1}$$

So we have one equation containing two unknown functions. Nevertheless, we can solve it and find the functions $Q(z\lambda)$ and $Q_0(z\lambda)$ in one-sided and two-sided problems, but, to this end, we need factorization of the function $1 - z\varphi(\lambda)$.

It is well known (see, e.g., [2]) that the factorization

$$1 - z\varphi(\lambda) = R_-(z,\lambda)R_+(z,\lambda), \quad |z| < 1, \quad \operatorname{Re}\lambda = 0, \tag{2}$$

holds, where the function $R_+(z,\lambda)$ is analytic with respect to λ in the left half-plane $\operatorname{Re}\lambda < 0$, continuous at the border, and it is bounded and does not equal zero when $\operatorname{Re}\lambda \le 0$. The function $R_-(z,\lambda)$ has similar properties in the right half-plane. The components of a factorization with the above properties are defined uniquely up to a constant factor. In addition, the functions $R_+(z,\lambda)$, $R_+^{-1}(z,\lambda)$ belong to $S\big([0,\infty)\big)$, and the functions $R_-(z,\lambda)$, $R_-^{-1}(z,\lambda)$ belong to $S\big((-\infty,0]\big)$. Here $S(A)$ denotes the set of functions g taking the form

$$g(\lambda) = \int_A e^{\lambda y}\,dG(y), \quad \text{where} \quad \int_A |dG(y)| < \infty, \quad \operatorname{Re}\lambda = 0.$$

Given a function $g \in S(\mathbb{R})$, we define

$$[g(\lambda)]^A = \int_A e^{\lambda y}\,dG(y)$$

for each Borel set A. As an example, we now show how the main identity (1) can be solved in one-sided and two-sided problems (see [3], [4]).

Theorem 2 *Let $b > 0$ and $B = [b,\infty)$. Then*

$$Q(z,\lambda) = R_+(z,\lambda)\big[R_+^{-1}(z,\lambda)\big]^{[b,\infty)}, \quad |z| < 1, \quad \operatorname{Re}\lambda = 0. \tag{3}$$

Proof. We use the relation

$$R_-(z,\lambda)Q_0(z,\lambda) = -R_-(z,\lambda) + R_+^{-1}(z,\lambda)\big(1 - Q(z,\lambda)\big),$$

which follows from (1) and (2). The left-hand side of the above relation belongs to $S\big((-\infty,b)\big)$, so

$$\Big[- R_-(z,\lambda) + R_+^{-1}(z,\lambda)\big(1 - Q(z,\lambda)\big)\Big]^{[b,\infty)} \equiv 0.$$

Clearly, $\big[R_-(z,\lambda)\big]^{[b,\infty)} \equiv 0$. Further, under our conditions, $Q(z,\lambda) \in S([b,\infty))$, so $R_+^{-1}(z,\lambda)Q(z,\lambda) \in S\big([b,\infty)\big)$. Hence,

$$\Big[R_+^{-1}(z,\lambda)\big(1 - Q(z,\lambda)\big)\Big]^{[b,\infty)} = \Big[R_+^{-1}(z,\lambda)\Big]^{[b,\infty)} - R_+^{-1}(z,\lambda)Q(z,\lambda) = 0.$$

A symmetric reasoning establishes that

$$Q(z,\lambda) = R_-(z,\lambda)\big[R_-^{-1}(z,\lambda)\big]^{(-\infty,a]} \quad \text{if } \ B=(-\infty,a], \quad a<0. \tag{4}$$

By definition, given a function $g \in S(\mathbb{R})$, we put

$$(\mathcal{L}_- g)(z,\lambda) = R_-(z,\lambda)\big[R_-^{-1}(z,\lambda)g(\lambda)\big]^{(-\infty,a]},$$

$$(\mathcal{L}_+ g)(z,\lambda) = R_+(z,\lambda)\big[R_+^{-1}(z,\lambda)g(\lambda)\big]^{[b,\infty)}.$$

Here $|z|<1$, $\operatorname{Re}\lambda = 0$, the function g may also depend on z. As can be seen from the definition, the operators $\mathcal{L}_\pm$ depend also on z, a, and b. For brevity, we do not emphasize this fact in the notations of operators. Put $e(\lambda)=e(z,\lambda)\equiv 1$. In the new notations, the formulas (3) and (4) can be rewritten in the following way:

$$Q(z,\lambda) = (\mathcal{L}_+ e)(z,\lambda) \ \text{ if } \ B=[b,\infty),$$

$$Q(z,\lambda) = (\mathcal{L}_- e)(z,\lambda) \ \text{ if } \ B=(-\infty,a].$$

It turns out that the double LST in the two-sided problem can be also expressed via operators $\mathcal{L}_\pm$. Really, put $B=(-\infty,a]\cup[b,\infty)$ then

$$N = \min\big\{n\ge 1: \ S_n \notin (a,\,b)\big\}, \quad a<0, \ b>0.$$

Let

$$Q_1(z,\lambda) = \mathbf{E}\big(z^N e^{\lambda S_N};\ S_N \le a\big), \quad Q_2(z,\lambda) = \mathbf{E}\big(z^N e^{\lambda S_N};\ S_N \ge b\big).$$

Then $Q(z,\lambda = Q_1(z,\lambda)+Q_2(z,\lambda)$.

In the same way as in Theorem 2, from (1) and (2) we obtain

$$Q_2(z,\lambda) = (\mathcal{L}_+ e)(z,\lambda) - (\mathcal{L}_+ Q_1)(z,\lambda), \tag{5}$$

$$Q_1(z,\lambda) = (\mathcal{L}_- e)(z,\lambda) - (\mathcal{L}_- Q_2)(z,\lambda). \tag{6}$$

Substituting the expression (6) for $Q_1(z,\lambda)$ into (5) leads to the identity

$$Q_2(z,\lambda) = (\mathcal{L}_+ e)(z,\lambda) - (\mathcal{L}_+\mathcal{L}_- e)(z,\lambda) + (\mathcal{L}_+\mathcal{L}_- Q_2)(z,\lambda),$$

and, in a similar way, we arrive at the identity for Q_1:

$$Q_1(z,\lambda) = (\mathcal{L}_- e)(z,\lambda) - (\mathcal{L}_-\mathcal{L}_+ e)(z,\lambda) + (\mathcal{L}_-\mathcal{L}_+ Q_1)(z,\lambda). \tag{7}$$

Further, for a random walk with nonzero drift, consider the random variable η equal to the number of upcrossings of the strip with boundaries at the levels $a<0$ and $b>0$. It turns out [5] that, in this case,

$$\mathbf{P}(\eta \ge k) = \lim_{z\to 1}\big((\mathcal{L}_+\mathcal{L}_-)^k e\big)(z,0), \quad k\ge 1.$$

Thus, we see that, in many boundary crossing problems connected with the achievement of a set with linear boundaries, LST of the distributions under study are expressed in terms of the operators $\mathcal{L}_{\pm}$. So, we need to clarify the probabilistic meaning of these operators, as well as the possibility of finding explicit expressions for them and asymptotic representations.

Discuss a probabilistic meaning. First, it is not difficult to deduce from (1) that $Q(z,\lambda) = 1 - R_+(z,\lambda)$ for $B = (0,\infty)$ and $Q(z,\lambda) = 1 - R_-(z,\lambda)$ for $B = (-\infty, 0)$. In both of these cases the function $Q(z,\lambda)$ is a joint distribution of the corresponding ladder epoch and ladder height of the random walk. Thus, using factorization components for finding the LST of distributions of boundary functionals means that the desired distributions are expressed in terms of the distributions of ladder values, which is quite natural.

Further, let $\tau \geq 0$ be an arbitrary stopping time, possibly improper. At the event $\{\tau < \infty\}$, we define the random variables

$$\tau_+(b) = \inf\{n \geq \tau:\ S_n \geq b\}, \quad \tau_-(a) = \inf\{n \geq \tau:\ S_n \leq a\}.$$

Suppose that the double transform $f(z,\lambda) = \mathbf{E}\big(z^{\tau}\exp\{\lambda S_{\tau}\};\ \tau < \infty\big)$ is known. The problem is to find the functions

$$f_+(z,\lambda) = \mathbf{E}\big(z^{\tau_+(b)}\exp\{\lambda S_{\tau_+(b)}\};\ \tau_+(b) < \infty\big),$$

$$f_-(z,\lambda) = \mathbf{E}\big(z^{\tau_-(a)}\exp\{\lambda S_{\tau_-(a)}\};\ \tau_-(a) < \infty\big).$$

The following assertion is obtained in [5].

Theorem 3 *For $|z| < 1$ and $\operatorname{Re}\lambda = 0$, the following relations hold:*

$$f_{\pm}(z,\lambda) = (\mathcal{L}_{\pm} f)(z,\lambda).$$

The assertion of this theorem makes clear the probabilistic meaning of all summands in (5) and (6). We note in passing that, under the conditions of the theorem, the distributions of jumps of a walk to the time τ and after it may not coincide. This makes it possible to consider random walks in which the distribution of jumps varies at the moment of passing certain boundaries.

Next, we discuss the possibilities of calculating the factorization components and operators $\mathcal{L}_{\pm}$ in an explicit form. The explicit form of the factorization components is known for Gaussian random walks [6] and for walks for which the function $\mathbf{E}\big(\exp\{\lambda X\};\ X < 0\big)$ or $\mathbf{E}\big(\exp\{\lambda X\};\ X > 0\big)$ is rational [2]. For example, if the function

$$\mathbf{E}\big(\exp\{\lambda X\};\ X > 0\big) = \frac{R(\lambda)}{P(\lambda)}, \quad \text{where} \quad P(\lambda) = \prod_{i=1}^{k}(\lambda - p_i),$$

is rational then

$$R_+(z,\lambda) = \frac{\Lambda(z,\lambda)}{P(\lambda)}, \qquad R_-(z,\lambda) = \frac{(1-z\varphi(\lambda))P(\lambda)}{\Lambda(z,\lambda)},$$

where $\Lambda(z,\lambda) = \prod_{j=1}^{k}(\lambda - \lambda_j(z))$, and $\lambda_1(z), \ldots, \lambda_k(z)$ are zeros of the function $1 - z\varphi(\lambda)$ in the right half-plane (with considering their multiplicities). In this case the calculation of $(\mathcal{L}_+ g)(z,\lambda)$ becomes a simple exercise if the function $R_+^{-1}(z,\lambda)$ is first decomposed on simple fractions.

Let us now investigate the asymptotic behavior of the operators $\mathcal{L}_\pm$ as $a \to -\infty$, $b \to \infty$. We assume here that the distribution of X contains an absolutely continuous component and the Cramér condition holds: $\varphi(\lambda) < \infty$ for $-\gamma \le \lambda \le \beta$, $\gamma > 0$, $\beta > 0$. In addition, we assume that $\mathbf{E}e^{\beta X} > 1$ if $\mathbf{E}X < 0$ and $\mathbf{E}e^{-\gamma X} > 1$ if $\mathbf{E}X > 0$. Under these conditions, one can distinguish the principal terms of the asymptotics for $(\mathcal{L}_\pm g)(z,\lambda)$ as $a \to -\infty$, $b \to \infty$ and estimate the remainders that turn out to be exponentially small in comparison with the principal terms (see [4]). As a result, for the two-sided boundary crossing problem, from (7) we obtain

$$\mathbf{E}\big(z^N e^{\lambda S_N}; S_N \ge b\big) = v_z(\lambda) e^{\lambda b} \frac{e^{-\lambda_+(z)b}(1 - v_2(z)\mu^a(z))}{1 - v_1(z)v_2(z)\mu^{a+b}(z)} \big(1 + O\big(e^{-\varepsilon b}\big)\big) \tag{8}$$

as $a \to -\infty$, $b \to \infty$, uniformly in $z \in (1-\delta, 1)$ for some $\delta > 0$, where

$$v_z(\lambda) = \frac{R_+(z,\lambda)}{(\lambda - \lambda_+(z))R'_+(z,\lambda_+(z))}, \qquad u_z(\lambda) = \frac{R_-(z,\lambda)}{(\lambda - \lambda_-(z))R'_-(z,\lambda_-(z))},$$

$$v_1(z) = v_z(\lambda_-(z)), \qquad v_2(z) = u_z(\lambda_+(z)), \qquad \mu(z) = \exp\{\lambda_-(z) - \lambda_+(z)\},$$

and $\lambda_-(z) < 0 < \lambda_+(z)$ are zeros of the function $1 - z\varphi(\lambda)$.

The corresponding assertion for the one-sided problem is a particular case of (8) with $a = -\infty$. If $\mathbf{P}(X \ge t) = q\exp\{-\alpha t\}$ for $t \ge 0$ then $v_z(\lambda) = \dfrac{\lambda_+(z) - \alpha}{\lambda - \alpha}$ and $O\big(e^{-\varepsilon b}\big)$ vanishes in (8).

The main terms of the resulting asymptotic representations are easily invertible in the variable λ. Tending z to 1, from them one can obtain many useful consequences. The inversion of the principal parts with respect to z is a sufficiently difficult task that is far beyond the scope of this article.

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