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Asymptotic Ruin Probabilities for a Multidimensional Renewal Risk Model with Multivariate Regularly Varying Claims

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Abstract. This paper studies a continuous-time multidimensional risk model with constant force of interest and dependence structures among random factors involved. The model allows a general dependence among the claim-number processes from different insurance businesses. Moreover, we utilize the framework of multivariate regular variation to describe the dependence and heavy-tailed nature of the claim sizes. Some precise asymptotic expansions are derived for both finite-time and infinite-time ruin probabilities.

Keywords: asymptotics, dependence, multidimensional renewal risk model, multivariate regular variation, ruin probability.

1. Introduction

Consider an insurance company which simultaneously operates d kinds of businesses. Its surplus process can be described by the following multidimensional risk model:

$$\begin{bmatrix} U_1(t) \\ \vdots \\ U_d(t) \end{bmatrix} = \begin{bmatrix} \rho_1 x \mathrm{e}^{rt} \\ \vdots \\ \rho_d x \mathrm{e}^{rt} \end{bmatrix} + \begin{bmatrix} c_1 \int_0^t \mathrm{e}^{r(t-s)} \mathrm{d}s \\ \vdots \\ c_d \int_0^t \mathrm{e}^{r(t-s)} \mathrm{d}s \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^{N_1(t)} X_{1i} \mathrm{e}^{r(t-\tau_{1i})} \\ \vdots \\ \sum_{i=1}^{N_d(t)} X_{di} \mathrm{e}^{r(t-\tau_{di})} \end{bmatrix}$$

$t \geq 0$, where $\{(U_1(t), \ldots, U_d(t))\,; t \geq 0\}$ denotes the multidimensional surplus process, $r \geq 0$ the constant force of interest, $(\rho_1 x, \ldots, \rho_d x)$ the vector of initial surpluses assigned to different businesses with positive ρ_1 , ..., ρ_d such that $\sum_{k=1}^{d} \rho_k = 1$, $(c_1, \ldots, c_d)$ the vector of constant premium rates, $\{(X_{1i}, \ldots, X_{di}); i \geq 1\}$ the sequence of claim-size vectors, and τ_{k1}, τ_{k2}, ... the claim-arrival times of the kth business with the corresponding claim-number process $\{N_k(t); t \geq 0\}$ for $k = 1, \ldots, d$.

Define the finite-time and infinite-time ruin probabilities corresponding to risk model (1) as

$$\psi(x; T) = \mathbb{P}\left(T_{\max} \leq T \,|\, (U_1(0), \ldots, U_d(0)) = (\rho_1 x, \ldots, \rho_d x)\right),$$

and

$$\psi(x) = \mathbb{P}\left(T_{\max} < \infty \middle| (U_1(0), \ldots, U_d(0)) = (\rho_1 x, \ldots, \rho_d x)\right),$$

where

$$T_{\max} = \inf\{t > 0 : \max\{U_1(t), \ldots, U_d(t)\} < 0\}$$

denotes the ruin time with $\inf \emptyset = \infty$ by convention. In this paper, we are concerned with the precise asymptotic expansions for both $\psi(x;T)$ and $\psi(x)$ as $x \to \infty$.

Recently, Yang and Li (2014) and Li (2015) introduced the Farlie–Gumbel–Morgenstern (FGM) dependence structure into claim sizes from different businesses in bidimensional risk models. Under certain technical conditions, they obtained asymptotic expansions for the finite-time and infinite-time ruin probability respectively. Although these works improved the previous ones to some extent, they can only be regarded as a beginning of this direction. For example, the FGM structure is so special that the problems under consideration and the corresponding treatment are essentially parallel to that in the independence case. Moreover, they also assumed that the two insurance businesses share a common claim-number process, which weakens the practicability of their results severely.

Contrasted to the prosperity of the study on bidimensional risk models, the asymptotic behavior for multidimensional models has been little investigated. Among the few contributions, Huang et al. (2014) considered a discrete-time multidimensional risk model, in which the claim sizes from different businesses follow a dependence structure given in terms of copulas. Assuming that the claim sizes are regularly varying, the authors derived asymptotic expansions for finite-time ruin probabilities. Recently, Li et al. (2015) studied a continuous-time multidimensional risk model, which is a special interest-free ($r = 0$) case of (1) with an identical Poisson claim-number process. They still only focused on finite-time ruin probabilities and obtained the corresponding asymptotic formulas for asymptotically independent regularly varying claim sizes.

The present paper is devoted to extend the existing works from three main aspects. First, we drop the restriction that all businesses have a totally identical claim-number process, and instead introduce a general dependence structure into the different claim-number processes. Second, we synthetically model the dependence and heavy-tailed nature of claim sizes by a unified framework of multivariate regular variation. Last but not the least, we obtain asymptotic expansions for both finite-time and infinite-time ruin probabilities.

2. Main section

2.1. Regular Variation

A distribution function $F = 1 - \overline{F}$ on $[0, \infty)$ is said to belong to the class of regular variation if $\overline{F}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \qquad y > 0,$$

holds for some $0 < \alpha < \infty$. We signify the regularity property in (2.1) as $\overline{F} \in \mathcal{R}_{-\alpha}$.

For a distribution function F with $\overline{F} \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, by Proposition 2.2.3 of Bingham et al. (1987) we know that, for any $0 < p_1 < \alpha < p_2 < \infty$ and $C > 1$, there is some $D > 0$ such that the inequalities

$$\frac{1}{C} \min\left\{y^{-p_1}, y^{-p_2}\right\} \leq \frac{\overline{F}(xy)}{\overline{F}(x)} \leq C \max\left\{y^{-p_1}, y^{-p_2}\right\}$$

hold whenever $x > D < xy$. We can derive from (2.1) that if $\overline{F} \in \mathcal{R}_{-\alpha}$ then, for any $p > \alpha$,

$$x^{-p} = o\left(\overline{F}(x)\right), \qquad x \to \infty.$$

It is known that a distribution function F with a regularly varying tail belongs to the long-tailed distribution class $\mathcal{L}$ characterized by $\overline{F}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x \to \infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = 1, \qquad y \in (-\infty, \infty).$$

We refer to Bingham et al. (1987) and Embrechts et al. (1997) for a nice review on related heavy-tailed distribution classes.

2.2. Dependence Assumptions

Consider the multidimensional risk model (1). Throughout, we assume that $\{(N_1(t), \ldots, N_d(t)); t \geq 0\}$ and $\{(X_{1i}, \ldots, X_{di}); i \geq 1\}$ are mutually independent.

For $k = 1, \ldots, d$, denote by $\theta_{k1} = \tau_{k1}$ and $\theta_{ki} = \tau_{ki} - \tau_{k,i-1}$ for $i = 2$, $3, \ldots$ the inter-arrival times of claims from the kth business. We introduce a general dependence structure into the different claim-number processes through the following assumption:

Assumption 2.1. $\{(\theta_1, \ldots, \theta_d), (\theta_{1i}, \ldots, \theta_{di}); i \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) nonnegative random vectors, but the d components of each vector can be arbitrarily dependent.

Clearly, under Assumption 2.1 all of $\{N_1(t); t \geq 0\}$, ..., $\{N_d(t); t \geq 0\}$ are traditional renewal processes, and they inherit dependences from that among θ_1, ..., θ_d. Further, for $(t_1, \ldots, t_d) \in [0, \infty)^d$, we write

$$N(t_1, \ldots, t_d) = \max\{i : \tau_{1i} \leq t_1, \ldots, \tau_{di} \leq t_d\} = \min_{1 \leq k \leq d} \{N_k(t_k)\}$$

and

$$\lambda(t_1, \ldots, t_d) = \mathbb{E}(N(t_1, \ldots, t_d)) = \sum_{i=1}^{\infty} \mathbb{P}(\tau_{1i} \leq t_1, \ldots, \tau_{di} \leq t_d),$$

which are called in the literature the d-dimensional renewal process and the corresponding renewal function respectively.

Before modeling the dependence structure among the claim sizes, we need to introduce the concept of multivariate regular variation (MRV) first. A random vector $(Z_1, \ldots, Z_d)$ taking values in $[0, \infty]^d \setminus \{\mathbf{0}\}$ is said to follow a distribution with a multivariate regularly varying tail if there exist some $0 < \alpha < \infty$, some distribution function F with $\overline{F} \in \mathcal{R}_{-\alpha}$, and some Radon measure ν on $[0, \infty]^d \setminus \{\mathbf{0}\}$ satisfying $\nu\left([0, \infty]^d \setminus \{\mathbf{0}\}\right) > 0$ such that the following vague convergence holds as $x \to \infty$:

$$\frac{1}{\overline{F}(x)} \mathbb{P}\left(\frac{(Z_1, \ldots, Z_d)}{x} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \qquad \text{on } [0, \infty]^d \setminus \{\mathbf{0}\}.$$

In this case, we write $(Z_1, \ldots, Z_d) \in \mathrm{MRV}(\alpha, F, \nu)$.

Now we model the dependences among the claim sizes from different businesses via the framework of MRV. Concretely speaking, we assume:

Assumption 2.2. $\{(X_1, \ldots, X_d), (X_{1i}, \ldots, X_{di}); i \geq 1\}$ is a sequence of i.i.d. nonnegative random vectors with $(X_1, \ldots, X_d) \in \mathrm{MRV}(\alpha, F, \nu)$ such that $\nu((\mathbf{1}, \boldsymbol{\infty}]) > 0$.

We can derive from Assumption 2.2 that

$$\nu(sK) = s^{-\alpha}\nu(K) \text{ for } s \in (0, \infty) \text{ and Borel set } K \subset [0, \infty]^d \setminus \{\mathbf{0}\},$$

$$\lim_{x \to \infty} \frac{1}{\overline{F}(x)} \mathbb{P}\left(\bigcap_{k=1}^{d} \{X_k > x\}\right) = \nu((\mathbf{1}, \boldsymbol{\infty}]) > 0,$$

and

$$\lim_{x \to \infty} \frac{\mathbb{P}(X_k > x)}{\overline{F}(x)} = \nu((\mathbf{1}_k, \boldsymbol{\infty}]) =: a_k > 0, \quad k = 1, \ldots, d,$$

where $\mathbf{1}_k$ is the vector with the kth element being 1 and the other elements being 0. Relation (2.2) indicates that the tails of $X_1, \ldots, X_d$ are regularly varying and mutually comparable. This fact, combined with (2.2), implies that $X_1, \ldots, X_d$ are pairwise asymptotically dependent. Additionally, we know from (2.2) and (2.2) that

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \mathbb{P}\left(\bigcap_{k=1}^{d} \{X_k > b_k x\}\right) = \nu\left((b_1, \infty] \times \cdots \times (b_d, \infty]\right) > 0$$

holds for any $(b_1, \ldots, b_d) \in [0, \infty]^d \setminus \{\mathbf{0}\}$. See Resnick (1987, 2007) for a comprehensive discussion on MRV and see Embrechts et al. (2009), Böcker and Klüppelberg (2010), Mainik and Rüschendorf (2010), Joe and Li (2011), and Tang and Yuan (2013) for its applications in finance and insurance.

We remark that the concept of MRV is developed from multivariate extreme value theory, which has been extensively applied to investigate extreme phenomena in finance and insurance. The framework of MRV provides us with an effective platform to quantify extreme risks, since, as analyzed above, it can model enormous sizes (heavy-tailed nature) of extreme risks and capture their tail dependences (asymptotic dependence) simultaneously.

In what follows, for any $(b_1, \ldots, b_d) \in [0, \infty]^d \setminus \{\mathbf{0}\}$, we will write

$$\nu\left((b_1, \infty] \times \cdots \times (b_d, \infty]\right) =: V\left(b_1, \ldots, b_d\right).$$

2.3. Main Results

Hereafter, all limit relations hold as $x \to \infty$ unless otherwise stated. For two positive functions f and g, we write $f(x) \lesssim g\ (x)$ or $g(x) \gtrsim f(x)$ if $\limsup f(x)/g(x) \leq 1$ and write $f(x) \sim g(x)$ if both $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$. To avoid triviality, a nonnegative random variable is always assumed to be nondegenerate at 0.

Now, we are ready to present our first main result for the finite-time ruin probability.

Theorem 2.1 *Consider risk model (1). Let Assumptions 2.1 and 2.2 hold. Then, for every T such that $\lambda(T, \ldots, T) > 0$, we have*

$$\psi\left(x, T\right) \sim \left[\int_{0-}^{T} \cdots \int_{0-}^{T} V\left(\rho_1 \mathrm{e}^{rt_1}, \ldots, \rho_d \mathrm{e}^{rt_d}\right) \lambda\left(\mathrm{d}t_1, \ldots, \mathrm{d}t_d\right)\right] \overline{F}\left(x\right).$$

Next, we focus on the infinite-time ruin probability. To this end, we naturally require $r > 0$ in risk model (1) for the convergence of quantities under consideration.

Theorem 2.2 *In addition to the other conditions of Theorem 2.1, if* $r > 0$ *then (2.1) holds also for* $T = \infty$*, i.e., we have*

$$\psi(x) \sim \left[\int_{0-}^{\infty} \cdots \int_{0-}^{\infty} V\left(\rho_1 \mathrm{e}^{rt_1}, \ldots, \rho_d \mathrm{e}^{rt_d}\right) \lambda\left(\mathrm{d}t_1, \ldots, \mathrm{d}t_d\right)\right] \overline{F}(x).$$

Particularly, if the businesses share a common claim-number process, i.e., $N_1(t) \equiv \cdots \equiv N_d(t) \equiv N(t)$, then we can immediately obtain more elegant and transparent forms for (2.1) and (2.2) by applying (2.2).

Corollary 2.1 *In addition to the other conditions of Theorem 2.1, if* $N_1(t) \equiv \cdots \equiv N_d(t) \equiv N(t)$ *with* $\lambda(t) = \mathbb{E}(N(t))$ *then*

$$\psi(x, T) \sim \left(\int_{0-}^{T} \mathrm{e}^{-\alpha rt} \lambda(\mathrm{d}t)\right) V(\rho_1, \ldots, \rho_d) \overline{F}(x).$$

Further if $r > 0$ *then*

$$\psi(x) \sim \frac{\mathbb{E}\left(\mathrm{e}^{-\alpha r\theta}\right)}{1 - \mathbb{E}\left(\mathrm{e}^{-\alpha r\theta}\right)} V(\rho_1, \ldots, \rho_d) \overline{F}(x),$$

where θ *is the generic random variable of the inter-arrival times of* $\{N(t)\}$.

Formulas (2.1)–(2.1) reveal that the ruin probabilities of risk model (1) with our dependence structures assume a form of some constant times $\overline{F}(x)$. Although the constants before $\overline{F}(x)$ are involved in general, the formulas enable us to easily conduct numerical estimates for the ruin probabilities; see, e.g., Section 5 of Tang and Yuan (2013) for some specific examples of the MRV with explicit Radon measures.

Additionally, in view of Yang and Li (2014) and Li (2015), if the claim sizes are asymptotically independent, then the decay rate of the ruin probabilities of risk model (1) should be $\left(\overline{F}(x)\right)^d$, which is much more rapid than $\overline{F}(x)$ as shown in (2.1)–(2.1). This fact indicates that using the framework of MRV to model the dependences among the claim sizes can prevent the underestimate of risks to a very great extent.

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