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Feynman-type local integration of stochastic PDE

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Abstract. This communication is devoted to an investigation of Gaussian quasi-measures and Feynman integration on infinite dimensional spaces with values in the octonion algebra, also their applications to solutions of PDEs including that of hyperbolic type and related stochastic processes.

Keywords: analytical methods in probability theory, computational methods, Feynman integral, hyperbolic PDE.

1. Introduction

It is well-known that Gaussian quasi-measures and Feynman integrals with values in the field of complex numbers are very important in mathematics and mathematical physics, especially, in quantum mechanics, quantum field theory and partial differential equations.

On the other hand, the Cayley-Dickson algebras $\mathcal{A}_r$ over $\mathbf{R}$ are natural generalizations of the complex field, where $\mathcal{A}_2 = \mathbf{H}$ denotes the quaternion skew field, $\mathcal{A}_3 = \mathbf{O}$ denotes the octonion algebra, $\mathcal{A}_0 = \mathbf{R}$, $\mathcal{A}_1 = \mathbf{C}$. PDEs with real or complex coefficients of order higher than two is frequently possible to represent as compositions of the first and second order PDEs, but generally with octonion or Cayley-Dickson coefficients. That is why quaternion and octonion algebras have found important applications in partial differential equations, mathematical physics, quantum field theory, hydrodynamics, industrial and computational mathematics, noncommutative geometry.

2. Feynman-type local integration of PDE

Let an operator B_j be realized as an elliptic PDO $\hat{B}_j$ of the second order on the Sobolev space $H^2(\mathbf{R}^{m_j})$ by real variables $x_{1+\beta_{j-1}},...,x_{\beta_j}$, where $m_0 = 0$, $\beta_0 = 0$, $\beta_j = m_0 + ... + m_j$, $m_j \in \mathbf{N}$ for each $j = 1, 2, ...$

We consider a PDO of the form

$$\hat{B} = -\sum_{j=1}^{m} a_j \hat{B}_j,$$

where $a_j = -c_j^2$ for each $1 \le j \le l$, while $a_j = c_j^2$ for each $l+1 \le j \le m$, where $c_j \in \mathcal{A}_r \setminus \{0\}$ for each $j = 1, ...m$, $c \in cl(W_{r,m})$, where $2 \le r$,

$0 \le l \le m$, $c = (c_1, ..., c_m)$. A domain $W_{r,m}$ is contained in $\mathcal{A}_r^m$ such that $W_{r,m} = V_r^m$, where $V_r = \{z \in \mathcal{A}_r : \ z = \rho \exp(M\gamma), \ M \in \mathcal{A}_r, \ Re(M) = 0, \ |M| = 1, \ 0 \le \gamma < \pi/4, \ \rho_1 \le \rho \le 1\}$, $0 < \rho_1 < 1$ is a marked constant, $Re(z) = (z + z^*)/2$ for $z \in \mathcal{A}_r$, $z = z_0 i_0 + z_1 i_1 + ... + z_{2^r-1} i_{2^r-1}$, while $z_0, ..., z_{2^r-1}$ are in $\mathbf{R}$, whilst $\{i_0, i_1, ..., i_{2^r-1}\}$ is the standard basis of the Cayley-Dickson algebra $\mathcal{A}_r$ over $\mathbf{R}$ such that $i_0 = 1$, $i_l^2 = -1$ and $i_l i_k = -i_k i_l$ for each $l \ne k$ with $1 \le l$ and $1 \le k$, where $z^* = \tilde{z} = z_0 i_0 - z_1 i_1 - ... - z_{2^r-1} i_{2^r-1}$. Let

$$\sigma^* f(x) = \sum_{j=1}^{m} \sigma_j^* f(x) \text{ and}$$

$$\sigma_j^* f(x) = \sum_{k=\beta_{j-1}+1}^{\beta_j} \psi_{k;j} \frac{\partial f(x)}{\partial x_k},$$

where $\psi_{k;j} \in \mathcal{A}_r$ for each k and j. Then the operator

$$\hat{S} = \frac{\partial}{\partial t} + \hat{B} + \sigma^* \tag{1}$$

is defined on the respective Sobolev space. We denote by $[B_j]$ a matrix corresponding to $\hat{B}_j$ and by B_j a linear operator $B_j : \mathbf{R}^{m_j} \to \mathbf{R}^{m_j}$ prescribed by its matrix $[B_j]$.

Let w be a stochastic process which may be generalized in $C(T, \mathcal{A}_{r,C}^m)$ induced from the standard Wiener process in the Euclidean space $\mathbf{R}^m$ by transformation with the operator $U = \bigoplus_{j=1}^m a_j^{1/2} B_j$. Let also $V(t,x) : [0,\infty) \times H \to \mathbf{C}$ be a Borel measurable function of $t \in \mathbf{R}$ and $x \in H$ such that V is bounded on each bounded subset in $[0,\infty) \times \mathcal{A}_{r,C}^m$, where $\mathcal{A}_{r,C}$ denotes the complexified Cayley-Dickson algebra $\mathcal{A}_r$, $H = \mathcal{A}_{r,C}^m$. Let μ be an $\mathcal{A}_{r,C}$-valued quasi-measure on $X = C_0(T,H)$ induced by w. Put

$$\nu(t) = \int_0^t V(\tau, w(\tau)) d\tau$$

for each $0 \le t < \infty$ and

$$K(x,t) = \tilde{\int}_X \{\exp(-\nu(t)|w(t) = x\} \frac{\mu_{tU,tp}(dx)}{\lambda_n(dx)} d\mu(w(t)),$$

where $\tilde{\int}_X f(w(t)) d\mu(w(t))$ is the extension of the Feynman integral to the Feynman-type operator on the space $\Psi(U)$ of test fast descending at infinity locally analytic functions.

Let $\hat{L} = \hat{S} + V(x)$ be a PDO, where the PDO $\hat{S}$ is given by Formula (1), where $V(t,x)$ is a bounded function satisfying a uniform Hölder condition everywhere on $[0,\infty) \times \mathcal{A}_{r,C}^m$ may be besides a regular subset.

Then there exists an operator $\hat{K}$ on a dense subspace $\mathcal{D}(\hat{K})$ in $H^{2,1}(\mathbf{R}^n \times \mathbf{R} \to \mathcal{A}_{r,C})$ such that $\hat{K}$ provides a generalized solution of the PDE $\hat{L}\hat{K} = 0$.

3. Conclusions

Similar generalizations are for infinite dimensional PDE in separable Hilbert spaces, when operators B_j are of trace class. Other types of hyperbolic PDE also can be studied.

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