

# Two-sided Bounds for the Convergence Rate of Markov Chains

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**Abstract.** We study inhomogeneous continuous-time weakly ergodic Markov chains on a finite state space and consider an approach to obtaining upper and lower bounds on the rate of convergence. Some specific queueing models are also discussed.

**Keywords:** finite continuous-time Markov chain, rate of convergence, birth and death process, ergodicity, bounds, queueing systems.

## 1. Main section

Studies of ergodic properties of finite inhomogeneous continuous-time Markov chain started in the 70-s of the last century, see for instance [6–8]. Since 1990-s main problems in this direction have been connected with obtaining quantitative estimates of the rate of convergence, see [19].

There are many papers dealing with upper bounds for the rate of convergence, see, for instance, [5, 13, 15, 16]. Usually, either the limiting mode is determined [1], or, on the contrary, the state probabilities are determined as functions of time  $t$  and initial conditions [4].

In [23] we presented an approach to finding sharp upper bounds in natural metrics via essential positivity of the reduced intensity matrix of a Markov chain. These bounds are sharp for nonnegative difference of the initial conditions of two probability distributions of a Markov chain. However, in a general situation the assumption of nonnegativity of this difference does not hold and the real lower bound of the rate of convergence can be essentially smaller. At the same time, lower bounds for the convergence rate are very important, because they give an opportunity to determine the time until which the accurate approximation of transient probability characteristics by the limiting probability characteristics is impossible.

As far as we know, these bounds were studied previously only in the papers [10–12, 19, 21] for birth-death processes via the notion of the logarithmic norm of an operator and the corresponding estimates. In this note we deal with the weak ergodicity of (as a rule, inhomogeneous) continuous-time Markov chains and formulate an algorithm for the construction of

lower bounds for the rate of convergence in the special weighted norms related to total variation.

Our first approach is more general, it is closely connected with the notion of the logarithmic norm and the corresponding bounds for the Cauchy matrix [9, 11, 19–22], and allows to obtain general upper and lower bounds for the rate of convergence for a countable state space and general initial conditions.

The second approach is based on the properties of nonnegative and essentially nonnegative matrices; it can be applied to the study of sharpness of upper bounds.

The concept of the logarithmic norm of a square matrix was developed independently in [2, 14] as a tool to derive error bounds in the numerical integration of initial-value problems for a system of ordinary differential equations (see also the survey papers [17, 18]). The logarithmic norm of an operator function and the corresponding bounds for solutions of differential equation in Banach space has been developed in [3]. One can find a detailed discussion in [5, 11].

The second approach was described in details in our recent paper [23].

Here we outline the algorithm of the study.

First, we consider the forward Kolmogorov system for the inhomogeneous Markov chain  $X(t)$  in the form

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}(t),$$

where  $A(t)$  is the transposed intensity matrix of the process. We can put  $p_0(t) = 1 - \sum_{i \geq 1} p_i(t)$  and obtain from (1) the reduced system in the form

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t),$$

where  $\mathbf{f}(t) = (a_{10}(t), a_{20}(t), \dots, a_{S0}(t))^T$ ,  $\mathbf{z}(t) = (p_1(t), p_2(t), \dots, p_S(t))^T$ ,

$$B(t) = \begin{pmatrix} a_{11}(t) - a_{10}(t) & a_{12}(t) - a_{10}(t) & \cdots & a_{1S}(t) - a_{10}(t) \\ a_{21}(t) - a_{20}(t) & a_{22}(t) - a_{20}(t) & \cdots & a_{2S}(t) - a_{20}(t) \\ a_{31}(t) - a_{30}(t) & a_{32}(t) - a_{30}(t) & \cdots & a_{3S}(t) - a_{30}(t) \\ \dots & & & \\ a_{S1}(t) - a_{S0}(t) & a_{S2}(t) - a_{S0}(t) & \cdots & a_{SS}(t) - a_{S0}(t) \end{pmatrix}.$$

All bounds on the rate of convergence to the limiting regime for  $X(t)$  correspond to the same bounds of the solutions of system

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t).$$

The main step is finding a transforming matrix  $D$  such that the matrix  $B^*(t) = DB(t)D^{-1} = (b_{ij}^*(t))_{i,j=1}^S$  is essentially nonnegative.

Consider the quantities  $h^*(t) = \max_j \sum_i b_{ij}^*(t)$ ,  $h_*(t) = \min_j \sum_i b_{ij}^*(t)$ , and  $g^*(t) = \max_j \sum_i (|b_{ii}^*(t)| + \sum_{j \neq i} b_{ji}^*(t))$ .

**Theorem.** Let  $X(t)$  be a finite inhomogeneous Markov chain. Let there exists a nonsingular matrix  $D$  such that  $B^*(t)$  is essentially nonnegative and

$$\int_0^\infty h^*(\tau) d\tau = -\infty.$$

Then  $X(t)$  is weakly ergodic and the following bounds hold:

$$\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \leq \exp \left\{ \int_0^t h^*(\tau) d\tau \right\} \|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D},$$

for any corresponding initial conditions  $\mathbf{p}^*(0)$ ,  $\mathbf{p}^{**}(0)$ , and

$$\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \geq \exp \left\{ \int_0^t h_*(\tau) d\tau \right\} \|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D},$$

if the initial conditions are such that  $D(\mathbf{z}^*(0) - \mathbf{z}^{**}(0)) \geq \mathbf{0}$ . Moreover,

$$\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D^*} \geq \exp \left\{ - \int_0^t g^*(\tau) d\tau \right\} \|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D^*}$$

for any corresponding initial conditions  $\mathbf{p}^*(0)$ ,  $\mathbf{p}^{**}(0)$ .

**Remark 1.** There are some classes of Markov chains for which it is possible to find sufficiently accurate bounds on the rate of convergence. Namely, we can do so for birth-death-catastrophe process in [21], for a class of Markovian queues with batch arrivals and group services in [22] and for their combinations in [24].

**Remark 2.** A finite birth-death process with constant rates of birth  $\lambda_k(t) = a$  and  $\mu_{k+1}(t) = b$  was considered in [9, 12], where both upper and lower sharp bounds were obtained. Namely, the corresponding  $D = D^*$ , sharp  $\beta_* = \beta^* = a + b - 2\sqrt{ab} \cos \frac{\pi}{S+1} \rightarrow (\sqrt{a} - \sqrt{b})^2$  as  $S \rightarrow \infty$ , and sharp  $g^* = a + b + 2\sqrt{ab} \cos \frac{\pi}{S+1} \rightarrow (\sqrt{a} + \sqrt{b})^2$  as  $S \rightarrow \infty$  were found, where  $\beta_* = \beta^* = -h^* = -h_*$ .

**Remark 3.** The obtained lower bounds find real applications to specific models of queueing systems and biology with a very fast entry into limit mode, as it was illustrated by the corresponding examples and plots in [21].

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