On application of Markovization method to calculation of reliability function*

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Abstract

Markovization method is used for analysis of a double redundant hot standby heterogeneous system. Explicit expression for the reliability function of such system is derived in terms of its Laplace transform. Its sensitivity analysis and limiting behavior under rare failures are also in our focus.

Keywords: reliability function, redundant systems, markovization method

1 Introduction and Motivation

Calculation of a system reliability function is one of the principal problems in reliability theory. It is well known that even for the simplest double redundant repairable system this function has not been analytically calculated yet in case when elements reliability and recovery functions have general distributions.

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In paper [1] multi-dimensional alternative processes have been used for studying the complex reliability systems. A system has been considered in case of no limitations on the number of repair facilities. When there is enough repair units, the components of the process describing the system behavior become independent allowing to calculate the system characteristics. However, in the case of limited number of repair facilities the problem of calculation of reliability function has not been solved yet, and it is considered in the current paper for the simplest case of a homogeneous hot standby system with only one repair unit.

On the other hand in series of works by B.V. Gnedenko, A.D. Solov'ev [2–4] and others it was shown that under "quick" restoration the system life time distribution becomes asymptotically insensitive to the shapes of its elements' life and repair time distributions and in scale of the system mean life time it tends to the exponential one.

In papers [5–7] a homogeneous cold standby double redundant system has been considered in the case when one of the input distributions (either of life or repair time lengths) is exponential. For these models explicit expressions for stationary probabilities have been obtained which show their evident dependence on the non-exponential distributions in the form of their Laplace-Stiltjes transforms. At that, the numerical investigations, performed in [8], show that this dependence becomes vanishingly small under "quick" restoration. However the problem of the reliability function calculation was not considered. In current paper we study the reliability function of a heterogeneous hot standby repairable system with exponential distribution of its components' life time and general distribution of their repair time.

The paper is organized as follows. In the next section the problem setting and some notations will be introduced. A Markovization method for obtaining the reliability function of the system is proposed in section 3. Section 4 deals with sensitivity analysis of the reliability function in case of "rare" failures. The paper ends with conclusion and some problems description.

2 The Problem setting and notations

Consider a heterogeneous hot double redundant repairable reliability system. Life times of components are exponentially distributed random variables (r.v.) with parameters α_1 and α_2 correspondingly. The repair times of components have absolute continuous distributions with cumulative distribution functions (c.d.f.) $B_k(x)$ $(k = 1, 2)$ and probability density functions (p.d.f.) $b_k(x)$ $(k = 1, 2)$ correspondingly. All life and repair times are independent. The "up" (working) states of each component and the system will be marked by 0 and the "down" (failed) state by 1 correspondingly.

Under considered assumptions the system behavior can be described by a random process, which takes values from the system state space $E = \{0,\ 1,\ 2,\ 3\}$, were 0 means that both components are in up states, $i\,(i = 1,\ 2)$ means that the i-th component is under repair, and the other one is in up state, and state 3 means that both components are in down state and therefore the system is in down state. At that the system states subset $E_0 = \{0,\ 1,\ 2\}$ represents its working (up) states, and the subset $E_1 = \{3\}$ represents the system failure (down) state. Denote by

- $J(t)$ a random process, describing the system behavior: $J(t) = i$ if the system is in the state i;
- $\alpha = \alpha_1 + \alpha_2$ the summary intensity of the system failure;
- $b_k = \int_0^\infty (1 - B_k(x))dx$ k-th element repair time expectations;
- $\beta_k(x) = (1 - B_k(x)^{-1})b_k(x)$ k-th element conditional repair intensity given elapsed repair time is x;

We are interesting in study of the system lifetime, which is the time T if at least one component is working. Thus the system lifetime T can be represented as $T = \inf\{t : J(t) \in E_1\}$ and the system reliability function as

$$R(t) = \mathbf{P}\{T \le t\} = \mathbf{P}\{J(\tau) \in T_0,\ \tau \in (0,t)\}. \tag{1}$$

3 Markovization method

In this section the Markovization principle is used for the problem solution. To realize it consider the two-dimensional Markov process $Z = \{Z(t),\ t \ge 0)\}$, with $Z(t) = (J(t), X(t))$ where $J(t)$ represents as before the system state, and $X(t)$ is an additional variable, which means the elapsed repair time of $J(t)$-th component at time t. The process phase space equals to $\mathcal{E} = \{0, (1,x), (2,x), 3\}$, which mean: 0 – both components are working, $(1,x)$ – the second component is working, the first one is failed and repaired, and its elapsed repair time equal to x, $(2,x)$ – the first component is working, the second one is failed and repaired, and its elapsed repair time equal to x, 3 – both components are failed, and therefore the system is failed. Appropriate probabilities are denoted by $\pi_0(t)$, $\pi_1(t;x)$, $\pi_2(t;x)$, $\pi_3(t)$.

The state transition graph of the system is represented in figure 1.

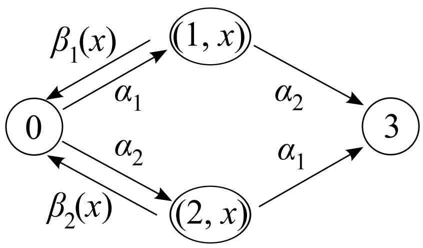

Figure 1: State transition graph of the system

By usual method of comparing the process probabilities in the closed times t and $t+\Delta$ the following Kolmogorov forward system of partial differential equations for these probabilities can be obtained,

$$\begin{aligned}
&\frac{d}{dt}\pi_0(t) = -\alpha\pi_0(t) + \int_0^t \pi_1(t,u)\beta_1(u)du + \int_0^t \pi_2(t,u)\beta_1(u)du,\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_1(t;x) = -(\alpha_2 + \beta_1(x))\pi_1(t;x),\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_2(t;x) = -(\alpha_1 + \beta_2(x))\pi_2(t;x),\\
&\frac{d}{dt}\pi_3(t) = \alpha_2\int_0^t \pi_1(t;u)du + \alpha_1\int_0^t \pi_2(t;u)du. \qquad (2)
\end{aligned}$$

jointly with the boundary and initial conditions

$$\pi_1(t,0) = \alpha_1\pi_0(t), \quad \pi_2(t,0) = \alpha_2\pi_0(t), \quad \pi_0(0) = 1. \qquad (3)$$

For the boundary condition explanation note that for $i = 1, 2$

$$\mathbf{P}\{J(t) = i,\, 0 \le X(t) \le \Delta\} = \pi_i(t;\theta\Delta)\Delta + o(\Delta) = \pi_0(t)(\alpha_i\Delta + o(\Delta)).$$

Theorem 1. The Laplace transform of the reliability function for considered system has the form

$$\tilde{R}(s) = \frac{(s+\alpha_1)(s+\alpha_2) + \alpha_1\phi_1(s) + \alpha_2\phi_2(s)}{(s+\alpha_1)(s+\alpha_2)(s+\psi(s))}, \qquad (4)$$

where for simplicity the following notations are used

$$\phi_i(s) = (s+\alpha_i)(1 - \tilde{b}_i(s+\alpha_{i^*})),\ (i = 1, 2) \qquad (5)$$

$$\psi(s) = \alpha_1(1 - \tilde{b}_1(s+\alpha_2)) + \alpha_2(1 - \tilde{b}_2(s+\alpha_1)), \qquad (6)$$

where $i^* = 2$ for $i = 1$ and vice versa.

Proof. Following to the characteristic method for first order partial differential equations solution [9] the solution of the second and third of equations (2) can be represented as follows

$$\pi_i(t,x) = h_i(t-x)e^{-\alpha_{i*}x}(1-B_i(x)), \tag{7}$$

where functions $h_i(x)$ should be find from boundary conditions (3). These solutions have a natural explanations in terms of renewal theory, where function $h_i(t)$ can be considered as renewal density of i-th component of the process J. Really, in order to the system to be in time t in the state (i,x) $(i=1,2)$, in which i-th element failed x time before time t, it should occurs in time $t-x$ in this state and do not leave it during time x, which means that the other element does not fail during this time and the repair of the failed element does not finish during this time.

Using these solutions and the boundary conditions (3) one can find the functions $h_i(t)$ in terms of $\pi_0(t)$,

$$\pi_1(t,0) = h_1(t) = \alpha_1\pi_0(t), \quad \pi_2(t,0) = h_2(t) = \alpha_2\pi_0(t). \tag{8}$$

This allow to find an exact solution for first equation of the system (2) in terms of LT. Really, substitution into the first equation of (2) the expressions for $\pi_i(t,x)$ from (7) gives the following result

$$\frac{d}{dt}\pi_0(t) = -\alpha\pi_0(t) + \sum_{1\le i\le 2}\int_0^t h_i(t-u)e^{-\alpha_{i*}u}b_i(u)du.$$

Passing in the last expression to LT and taking into account that in initial time the system is in up state 0, which means that $\pi_0(0)=1$ gives

$$s\tilde{\pi}_0(s) - 1 = -\alpha\tilde{\pi}_0(s) + \tilde{h}_1(s)\tilde{b}_1(s+\alpha_2) + \tilde{h}_2(s)\tilde{b}_2(s+\alpha_1). \tag{9}$$

Substitution of the expressions for LT of the functions $h_i(.)$, obtained from (8) to the equation (9) and taking into account that $\alpha = \alpha_1 + \alpha_2$ allows to find $\tilde{\pi}_0(s)$ in the form

$$\begin{aligned} \tilde{\pi}_0(s) &= \left[s + \alpha_1(1-\alpha_1\tilde{b}_1(s+\alpha_2)) + \alpha_2(1-\tilde{b}_2(s+\alpha_1))\right]^{-1} = \\ &= \left[(s+\psi(s)\right]^{-1}. \end{aligned} \tag{10}$$

This allows also to find LT for functions $\pi_i(t)$ $(i=1,2)$ in terms of $\tilde{\pi}_0(s)$,

$$\tilde{\pi}_1(s) = \tilde{h}_1(s)\frac{1-\tilde{b}_1(s+\alpha_2)}{s+\alpha_2} = \alpha_1\frac{1-\tilde{b}_1(s+\alpha_2)}{s+\alpha_2}\tilde{\pi}_0(s), \tag{11}$$

$$\tilde{\pi}_2(s) = \tilde{h}_2(s)\frac{1-\tilde{b}_2(s+\alpha_1)}{s+\alpha_1} = \alpha_2\frac{1-\tilde{b}_2(s+\alpha_1)}{s+\alpha_1}\tilde{\pi}_0(s). \tag{12}$$

Taking into account that

$$\pi_3(t) = \mathbf{P}\{Z(t) = 3\} = \mathbf{P}\{T \le t\} = 1 - R(t)$$

for the system reliability function calculation we need to find the probability $\pi_3(t)$ of the absorbing state 3. We will calculate it in terms of LT. Taking into account the representation (7) for functions $\pi_i(t,x)$ the last of equations (2) can be represented in the form

$$\begin{aligned} \frac{d}{dt}\pi_3(t) = \alpha_1 \int_0^t \pi_2(t;u)du + \alpha_2 \int_0^t \pi_1(t;u)du = \qquad (13) \\ = \alpha_1 \int_0^t h_2(t-u)e^{-\alpha_1 u}(1 - B_2(u))du + \\ + \alpha_2 \int_0^t h_1(t-u)e^{-\alpha_2 u}(1 - B_1(u))du = \\ = \alpha_1\alpha_2 \int_0^t \pi_0(t-u)\left[e^{-\alpha_1 u}(1 - B_2(u)) + e^{-\alpha_2 u}(1 - B_1(u))\right] du. \end{aligned}$$

The LT of this equation gives the following representation for LT of probability of absorbing state 3

$$s\tilde{\pi}_3(s) = \alpha_1\alpha_2\tilde{\pi}_0(s)\left[\frac{1 - \tilde{b}_1(s+\alpha_2)}{s+\alpha_2} + \frac{1 - \tilde{b}_1(s+\alpha_1)}{s+\alpha_1}\right]. \qquad (14)$$

At least substitution into this equality expression (10) for $\tilde{\pi}_0(s)$ after some cumbersome algebraic calculations using the notations (5, 6) gives

$$\begin{aligned} s\tilde{\pi}_3(s) &= \frac{\alpha_1\alpha_2}{(s+\alpha_1)(1-\alpha_1)(1-\tilde{b}_1(s+\alpha_2)) + \alpha_2(1-\tilde{b}_2(s+\alpha_1))} \times \\ &\times \left(\frac{1 - \tilde{b}_1(s+\alpha_2)}{s+\alpha_2} + \frac{1 - \tilde{b}_2(s+\alpha_1)}{s+\alpha_1}\right) = \\ &= \frac{\alpha_1\alpha_2(\phi_1(s) + \phi_2(s))}{(s+\alpha_1)(s+\alpha_1)(s+\psi(s))}. \qquad (15) \end{aligned}$$

At least for LT of $R(t)$ after some additional algebraic calculations one has

$$\tilde{R}(s) = \frac{1}{s} - \tilde{\pi}_3(s) = \frac{1}{s} - \frac{\alpha_1\alpha_1\tilde{\pi}_0(s)}{s}\left[\frac{1 - \tilde{b}_1(s+\alpha_2)}{s+\alpha_2} + \frac{1 - \tilde{b}_2(s+\alpha_1)}{s+\alpha_1}\right]$$

and therefore

$$\begin{aligned}
\tilde{R}(s) &= \frac{1}{s}\left(1-\alpha_1\alpha_2\tilde{\pi}_0(s)\left[\frac{1-\tilde{b}_1(s+\alpha_2)}{s+\alpha_2}+\frac{1-\tilde{b}_2(s+\alpha_1)}{s+\alpha_1}\right]\right)= \\
&= \frac{1}{s}\left(1-\frac{\alpha_1\alpha_2[\phi_(s)+\phi_2(s)]}{(s+\alpha_1)(s+\alpha_2)\psi(s)}\right)= \\
&= \frac{(s+\alpha_1)(s+\alpha_2)+\alpha_1\phi_1(s)+\alpha_2\phi_2(s)}{(s+\alpha_1)(s+\alpha_2)(s+\psi(s))}. \qquad (16)
\end{aligned}$$

■

Thorough analysis of the theorem proof allows to propose the following Corollary.

Corollary 1. The LT $\tilde{\pi}_i(s)$, $(i\in\{0,1,2\})$ of the time dependent probabilities $\pi_i(t)$, $(i\in\{0,1,2\})$ are

$$\begin{aligned}
\tilde{\pi}_0(s) &= \frac{1}{s+\psi(s)}, \\
\tilde{\pi}_1(s) &= \alpha_1\frac{1-\tilde{b}_1(s+\alpha_2)}{(s+\alpha_2)(s+\psi(s))}, \\
\tilde{\pi}_2(s) &= \alpha_2\frac{1-\tilde{b}_2(s+\alpha_1)}{(s+\alpha_1)(s+\psi(s))},
\end{aligned}$$

where notations (5), (6) are used.

Moreover Theorem 1 and Corollary 1 allow to calculate time spent by the process in its transient states and mean time to failure, in particular, the following corollary holds.

Corollary 2. The mean life time of the system under consideration equals to

$$m=\mathbf{E}[T]=\tilde{R}(0)=\frac{\alpha_1\alpha_2+\alpha_1(1-\tilde{b}_1(\alpha_2))\alpha_2(1-\tilde{b}_2(\alpha_1))}{\alpha_1\alpha_2[\alpha_1(1-\tilde{b}_1(\alpha_2))+\alpha_2(1-\tilde{b}_2(\alpha_1))]}. \qquad (17)$$

Remark. It is necessary to note that for homogeneous case, when both intensities and c.d.f. coincide, the sum of probabilities $\pi_1(\cdot)$ and $\pi_2(\cdot)$ represents the probability that one of the two elements is in down state. In this case the results take the following form

$$\begin{aligned}
\tilde{\pi}_0(s) &= [s+2\alpha(1-\tilde{b}(s+\alpha))]^{-1}, \\
\tilde{\pi}_1(s)+\tilde{\pi}_2(s) &= 2\alpha\frac{1-\tilde{b}(s+\alpha)}{s+\alpha}\pi_0(s), \\
\tilde{\pi}_3(s) &= 2\alpha^2\frac{1-\tilde{b}(s+\alpha)}{s(s+\alpha)}\pi_0(s).
\end{aligned}$$

At that

$$\tilde{R}(s) = \frac{s+\alpha+2\alpha(1-\tilde{b}(s+\alpha))}{(s+\alpha)[s+2\alpha(1-\tilde{b}(s+\alpha))]} \tag{18}$$

that coincides with the results obtained before in [5–7].

Moreover for the Markov case, when $b(t) = \beta e^{-\beta t}$, these formulas after substitution of appropriate expressions for $\tilde{b}(s+\alpha) = \beta(s+\alpha+\beta)^{-1}$ coincide with appropriate expressions, obtained with the usual Markov approach

$$\begin{aligned}
\tilde{\pi}_0(s) &= [s+2\alpha(1-\tilde{b}(s+\alpha))]^{-1} = \frac{s+\alpha+\beta}{s^2+(3\alpha+\beta)s+2\alpha^2}, \\
\tilde{\pi}_1(s)+\pi_2(s) &= \alpha\frac{1-\tilde{b}(s+\alpha)}{s+\alpha}\tilde{\pi}_0(s) = \frac{2\alpha}{s^2+(3\alpha+\beta)s+2\alpha^2}, \\
\tilde{\pi}_3(s) &= \frac{2\alpha^2\pi_0(s)}{s}\frac{1-\tilde{b}(s+\alpha)}{s+\alpha} = \frac{2\alpha^2}{s(s^2+(3\alpha+\beta)s+2\alpha^2)} \\
\tilde{R}(s) &= \frac{1}{s} - \tilde{\pi}_3(s) = \frac{(s+(3\alpha+\beta)}{(s^2+(3\alpha+\beta)s+2\alpha^2)}.
\end{aligned}$$

4 Sensitivity analysis

The above formulas demonstrate an evident dependence of the reliability function on the shape of the repair time distribution. It is expressed in the form of Laplace-Stiltijes transformation of the repair time distribution in points of failure items intensities. Of course in the Markov case with exponential repair time distribution these dependencies are expressed in terms of only one parameter of the exponential distribution.

From another side it is very known that in the case of quick restoration the reliability function tends to exponent for any repair time distribution. In this section we consider the reliability function behavior instead of "quick" restoration under the "rare" failures.

For the model under consideration under rare failures it is understood the slow intensity of failures with respect to the fixed repair times. Thus we will suppose that

$$q = \max\{\alpha_1, \alpha_2\} \to 0.$$

Naturally the asymptotic analysis should be done with respect to some scale parameter. As such parameter the assymptotic mean lifetime value will be considered. Accordingly to (16) taking into account that under $q \to 0$ with $b_i = \mathbf{E}[B_i] = -\tilde{b}_i'(0) = \int_0^\infty (1-B_i(x))dx$ and $\rho_i = \alpha_i b_i$ one can find

$$\phi_i(0) = \alpha_i^2(1-\tilde{b}_i(\alpha_{i^*})) = \alpha_i^2 b_i \alpha_{i^*}$$

it takes the value

$$m = \mathbf{E}[T] = \tilde{R}(0) \quad = \quad \frac{\alpha_1\alpha_2 + \alpha_1\phi_1(0) + \alpha_2\phi_2(0)}{\alpha_1\alpha_2\psi(0)} = $$
$$= \quad \frac{1 + \rho_1 + \rho_2}{\alpha_1\alpha_2(b_1 + b_2)}.$$

Theorem 2. The reliability function of the considered model in scale $m = \mathbf{E}[T]$ has exponential distribution,

$$\lim_{q \to 0} \mathbf{P}\left\{\frac{T}{m} > t\right\} = e^{-t}.$$

Proof. For simplicity instead of big parametger m consider small parameter $\gamma = m^{-1}$. Consider asymptotic of the LT of function

$$R\left(\frac{t}{\gamma}\right) = \mathbf{P}\{\gamma T > t\}$$

under $\gamma \to 0$. For it consider asymptotic of its Laplace transform $\gamma\tilde{R}(\gamma s)$

$$\begin{aligned} \gamma\tilde{R}(\gamma s) &= \gamma\frac{(\gamma s + \alpha_1)(\gamma s + \alpha_2) + \alpha_1\phi_1(\gamma s) + \alpha_2\phi_2(\gamma s)}{(\gamma s + \alpha_1)(\gamma s + \alpha_2)(\gamma s + \psi(\gamma s))} = \\ &= \gamma\frac{1 + \rho_1 + \rho_2}{\gamma s(1 + \rho_1 + \rho_2) + \alpha_1\alpha_2(b_1 + b_2)} = \gamma\frac{1}{\gamma s + \gamma} = \\ &= \frac{1}{s + 1}. \end{aligned} \tag{19}$$

From here it follows that under $\gamma \to 0$

$$\mathbf{P}\{\gamma T > t\} = R\left(\frac{t}{\gamma}\right) \to e^{-t}.$$

■

5 Conclusion

The problem of reliability function investigation for a heterogeneous hot double redundant repairable system with exponentially distributed life times of its components has been considered. Explicit representation of this function in terms of Laplace transform has been found. It was shown that in case of rare failures of system elements the reliability function in scale of its mean life time is approximated with exponent.

The problem of the reliability function investigation for more complex systems and under more general conditions is open.

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