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Some Algorithms of Record Generation

A.I. Pakhteev

Institute of Physics, Mathematics and Information Technology,
Immanuel Kant Baltic Federal University,
A.Nevskogo 14, Kaliningrad, 236041 Russia

Abstract. In the present work, we discuss algorithms of record generation when records are taken from a gamma population.

Keywords: records, gamma-distribution, rejection and inverse-transform methods.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of random variables defined on a common probability space. Let us introduce the sequences of record times $L(n)$ and record values $X(n)$ as follows:

$$L(1) = 1,$$

$$L(n+1) = \min\left\{j : j > L(n),\, X_j > X_{L(n)}\right\}, \quad X(n) = X_{L(n)} \quad \text{for } n \geq 1.$$

The theory of records is much discussed and well developed; see, for example, [1]. A new and interesting turn in this theory is associated with simulation of record data; see the works [2], [3], [4], [5], [6] and [7].

The simulation of records is important for modeling such experiments where only record data is available. The concept of record sample was introduced in [8]. One situation where such samples arise is in industrial stress and life-testing wherein measurements are made sequentially and only values larger than all previous values are recorded. For some other examples, see [9], [10] and [11]. It should also be noted that record samples are used in statistics in data reduction procedures.

The Gamma distribution is important for statistics and simulation. It belongs to the exponential family, and many research methods applicable to the Gamma distribution are also applicable to the other distributions of this family. Generating Gamma random variables, in its turn, is an interesting simulation problem.

The goal of the present work is to develop and discuss generation algorithms of Gamma records. Generation algorithms of Gamma records are proposed in our work first time. The corresponding algorithms are based on the rejection method. Before presenting these algorithms we introduce the direct method of record generation by which records from any population can be generated. However, if a large number of weak records is needed this method is computationally burdensome and slow.

The direct method. *The value* $X(1) = X_1$ *is generated and kept. For* $n \geq 1$*, one can apply the recursive approach, which assumes that* $X(n)$ *is already obtained. One then generates observations* X_i *till one of them, say* X_j*, is greater than* $X(n)$*. Then* $X(n+1) = X_j$ *becomes the next record value, which is also kept.*

Let in the following $X_1, X_2, \ldots$ be independent and identically distributed random variables with a Gamma distribution $F(x \mid \alpha)$, where $\alpha > 0$ is a shape parameter. Let also $f(x \mid \alpha)$ be the corresponding density function, i.e.

$$f(x \mid \alpha) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} \quad (x, \alpha > 0).$$

It is known that the sequence $X(1), X(2), \ldots$ forms a Markov chain and

$$f_{X(n+1)|X(n)}(x_{n+1} \mid x_n, \alpha) = \frac{f(x_{n+1} \mid \alpha)}{1 - F(x_n \mid \alpha)} \quad (x_{n+1} > x_n),$$

where $F(x_n \mid \alpha) = \int_0^{x_n} f(u \mid \alpha)du$. In the following we will use (1) for modeling Gamma records. In the case of Gamma distribution the inverse function F^{-1} cannot be obtained explicitly. One can apply here either tables of the inverse functions F^{-1} or the rejection method. In our work, the algorithms of generation of record values will be based on the rejection method. In our talk we compare algorithms based on the rejection method and algorithms based on tables of F^{-1}. This comparison shows that the algorithms based on the rejection method are more efficient.

2. Simulation algorithms

2.1. Simulation algorithm of $X(n)$ for $\alpha \in (0,1)$

Algorithm: The sequence $X(n)$ $(n \geq 1)$ can be generated in accordance with the following algorithm.

STEP 1: Generate $X(1) = X_1$ with $F(x \mid \alpha)$ by the rejection method. For $n \geq 1$, apply the rejection method and the following recursive approach. Assume that $X(n) = x_n$ is already obtained.

STEP 2: Generate a random number $U_1 = u_1$. Generate $Y = y$ with

$$g(y \mid x_n) = e^{-y+x_n} \quad (y > x_n),$$

namely: $y = x_n - \log u_1$.

STEP 3: Generate a random number $U_2 = u_2$. If $u_2 < \left(\frac{y}{x_n}\right)^{\alpha-1}$, set $X(n+1) = y$. Otherwise, return to STEP 2.

Remark: One can show that $c^* = c^*(x_n) \to 1$ as $x_n \to \infty$.

Since for Gamma records $X(n) \overset{\text{a.s.}}{\to} \infty$, Remark 2.1 points out that Algorithm 2.1 (which is based on the rejection method) works eventually as an algorithm based on the inverse-transform method. That is that with time almost every generation in a generation experiment is accepted and becomes a new record. That is Algorithm 2.1 is effective and speedy.

2.2. Simulation algorithm of $X(n)$ for $\alpha > 1$

Algorithm: The sequence $X(n)$ $(n \geq 1)$ can be generated as follows.

STEP 1: Generate $X(1) = X_1$ with $F(x \mid \alpha)$ by the rejection method. For $n \geq 1$, apply the rejection method and the following recursive approach. Assume that $X(n) = x_n$ is already obtained.

STEP 2: Generate a random number $U_1 = u_1$. Generate $Y = y$ with

$$g(y \mid x_n, \mu_A^*) = \frac{1}{\mu_A^*} e^{-\frac{y - x_n}{\mu_A^*}} \quad (y > x_n),$$

where $\mu_A^* = \frac{2x_n}{x_n - \alpha + \sqrt{(x_n - \alpha)^2 + 4x_n}}$. The last generation can be done as $y = x_n - \mu_A^* \log u_1$.

STEP 3: Generate a random number $U_2 = u_2$. If

$$u_2 < \left(\frac{y\left(1 - \frac{1}{\mu_A^*})\right)}{\alpha - 1} \right)^{\alpha - 1} e^{-y\left(1 - \frac{1}{\mu_A^*}\right) + \alpha - 1},$$

set $X(n+1) = y$. Otherwise, return to STEP 2.

Remark: One can show that $c_A^* = c_A^*(x_n) \to 1$ as $x_n \to \infty$.

Remark 2.2 also indicates that Algorithm 2.2 works with time as an algorithm based on the inverse-transform method. That is Algorithm 2.2 is also effective and speedy.

3. Conclusions

We proposed two algorithms of record generation when records were taken from a Gamma population with shape parameter $\alpha < 1$ and $\alpha > 1$, respectively. By analyzing their asymptotic performance, we showed that though these algorithms are based on the rejection method, they worked with time as algorithms based on the inverse-transform method, i.e. they are speedy and effective. They are more effective than the algorithms based on the tables of inverse distribution.

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