UDC 51(091)

Mathematical Methods in Practical Mechanics: From Heron of Alexandria to Galileo

E. Zaytsev*

* *Department of History of Mathematics*
S.I. Vavilov Institute for the History of Science and Technology
Baltiiskaya 14, Moscow, Russia.

Abstract. The paper is devoted to the history of mathematical methods in practical mechanics used, in particular, for the description of motion of "simple machines." It is shown that the wide application of these methods has begun in the 16$^{\text{th}}$ c. after the introduction of counterweight in lifting technology.

Keywords: Mathematical methods in mechanics, history of pre-classical mechanics, simple machines, hoist, Heron of Alexandria, Galileo.

1. Introduction

The opposition between dynamics and statics is nowadays rather loose, as both disciplines are governed by the same laws of classical mechanics (in statics these laws are operative within the principle of virtual work).

In pre-classical mechanics, to the contrary, dynamics and statics were strictly separated due to the general idea about the absolute opposition between motion and rest. Now and then, attempts were made to bridge the gap between the two disciplines by reducing the rules of statics to the laws of Aristotelian dynamics (Physics V, 7), as e.g., in the science of weights developed in the school of Jordanus Nemorarius (13$^{\text{th}}$ c.). But these attempts were by no means successful. A short revival of this tactics in the middle of the 16$^{\text{th}}$ c. proved to be theoretically unproductive as well.

In the second half of the 16$^{\text{th}}$ c., a different approach emerged within which statics assumed the role of the theoretical foundation of dynamics. Now, the law of inverse proportionality (of weights and arms on scales in equilibrium) was applied to the movements of the so called "simple machines" (lever, wheel and axle, compound pulley, inclined plane, wedge and screw). The emergence of this approach marked the turning point in the development of theoretical mechanics that from that time on has got as its main objective the description of motion in mathematical terms.

2. Practical mechanics in antiquity

In techniques, in contrast to nature, the opposition between motion and rest is not absolute. A key example is the wheel which while rotating around the fixed axle exhibits both of the conflicting features. Describing its motion, the author of the pseudo-Aristotelian "Mechanical Problems" wrote: "it is a very great marvel that contraries should be present together, and the circle is made up of contraries. For to begin with, it is formed by motion and rest, things which are by nature opposed to one another." [1, 847b 18-21].

"Mechanical problems" (3^{rd} c. BC) is the earliest surviving work on practical mechanics, being at the origin of a series of special treatises on the subject including: the 10^{th} Book of "On architecture" by Vitruvius (1st c. BC), works on mechanics by Heron of Alexandria (1st c. AD) and the 8^{th} Book of "Mathematical Collection" by Pappus of Alexandria (3^{rd} c. AD).

"Mechanical problems" are characterized by two main features (i) the reduction of the majority of technical (artificial) motions to that of the lever, reduced in turn to the balance, and finally to the circular motion; and (ii) the utter disregard for quantitative parameters of these motions. The only exception is question 3 which constitutes an attempt to describe the movement of the lever in terms of inverse proportionality between the forces and the arms of their application. The author asks: "Why is it that . . . the exercise of little force raises great weights with the help of a lever . . . ?" His answer runs as follows. "Does the reason lie in the fact that . . . . the lever acts like the beam of a balance with the cord attached below and divided into two unequal parts? . . . As the weight moved is to the weight moving it, so, inversely, is the length of the arm bearing the weight to the length of the arm nearer to the power. The further one is from the fulcrum, the more easily will one raise the weight . . . " [1, 850b1-3].

The fact that the fragment is concise and totally isolated within the treatise suggests that the treatise was not aimed at the quantitative analysis of technical motion. The same is true of the other ancient texts on mechanics in which one finds neither calculations, nor numerical data with regard to that kind of motion; they contain qualitative estimations only. Thus, of the lever it is customarily noticed that the larger the leverage of the driving force application, the greater will be the gain in force. Similarly, of the wheel and the axle it is indicated that the gain in lift increases following the augmentation of the ratio of the diameter of the wheel to the diameter of the axle. Concerning the compound pulley (polyspast), it is noted that the increase in lift is due to the increase in the number of blocks. No specification in terms of numbers is usually given.

One exception is, however, known. This is "Mechanics" by Heron of Alexandria, more specifically, its second, theoretical part [2]. In it, Heron did attempt to quantify the movements of the "simple machines" by having derived the laws of their motions from the rule of equilibrium of weights on scales and using the theory of proportions.

In his work, Heron has established mathematical relations between two kinds of forces. One of them is that of the worker who drives the machine, while the other is the force of gravity that resists its motion. The former force is animal and active, while the latter is mechanical and passive. While having expressed the relations between the two forces in terms of a mathematical ratio, Heron first ensured their homogeneity by equating the effort of the worker to the weight that could be lifted by him without using technical devices, viz. to 5 talents (about 130 kg.). The identification of the vital force of the worker with that of the weight made them both partaking to a common genus — that of gravity (otherwise, they could not be considered as terms of a mathematical ratio).

Heron successfully reduced the movements of the lever, of the block, and of the wheel and the axle to the law of equilibrium and correctly solved the problems by using the model of two concentric circles. In the case of compound pulley and wedge, the reduction of motions to static conditions has not been, in fact, achieved. While having correctly defined the numerical ratios between the lifting force and the lifted weight (in the case of the compound pulley) and the force of blow and its effect (in the case of the wedge), he did not give any details of the reductive procedure. Dealing with another problem, that of finding the force needed to move a ball up an inclined plane, Heron did apply the proclaimed tactics (using the equilibrium conditions between the parts of the ball on the either sides of its vertical section passing through the point of its tangency with the plane), but he failed to provide the solution of the problem in terms of the specific ratio.

Although Heron's attempts to quantify the motions of the "simple machines" look rather impressive from the standpoint of classical mechanics, in antiquity they found — as far as we know — no response. It is worth to mention, that Pappus while reproducing the descriptions of the "simple machines" borrowed from the practical part of Heron's treatise, totally neglected the material of its theoretical (mathematical) part.

All in all, the contents of the surviving sources testify that the idea of mathematical treatment of technical motions was alien to ancient mechanics. The same can be said about its medieval offspring.

3. Practical mechanics in the 16th century

Fundamentally different in this respect was the practical mechanics of the second half of the 16th c. Treatises compiled at that time paid a considerable attention to the mathematical aspects of the motions of technical devices. This tendency culminated in the early treatise "Mechanics" by Galileo (about 1593). In it Galileo exposed methods for calculating the ratios of forces for all of the "simple machines", based on the law of equilibrium of weights [3].

Remarkable is, that this novel approach to the description of technical motion has originated not in the work by a theorist. A prominent role in its formation was played by G. del Monte, whose practical manual "Mechanica" was published in 1577, some fifteen years before Galileo started his scientific carrier. Beside the theorists, like Galileo and G. Benedetti, the manual of del Monte has strongly influenced more practically oriented authors as, for example, B. Lorini ("On Fortifications", 1593) and B. Baldi (Commentary on "Mechanical Problems", 1621). The application of quantitative estimations in practical mechanics seems to go back to Leonardo da Vinci.

Now, the following question arises: what changes occurred in practical mechanics in the 16^{th} century that might have triggered the birth of the idea of mathematizing of technical movements?

In order to answer it, let us turn to the hoisting machines. The main feature of ancient and medieval hoists was their reliance on muscular power. It was either a manual effort applied to ropes, levers, wheels, pulleys, etc., or a foot effort applied to treadwheels by workers walking from step to step. Within such a practice, it was hardly possible to produce a movement obeying strict numerical ratios. The first hindrance was the above-mentioned heterogeneity of the forces: that of the human power setting the hoists in motion, and that of the mechanical force of gravity resisting it. To establish a mathematical relation, one of the forces must have been first identified with the other (in one way or another). In most cases, it was the mechanical force that was treated as the animal one. Within this practice, relations between the forces could be described in qualitative terms only, that is, "greater than", "less than", or "equal".

The next hindrance was — to use a modern parlance — a non-additivity of animal forces and consequently of the weights (as reduced to them). The joint action of the workers, e.g., pulling a weight, was greater than the sum of their individual efforts (this property, odd as it seems from the viewpoint of classical mechanic, was repeatedly emphasized by pre-classical authors) [4]. The non-additivity of manual forces can be in part explained by the fact that while lifting a heavy load, the workers try to maximize efforts and synchronize actions, rather than making them smooth and step by step cumulative (which is a prerequisite for mathematical description of motion).

The situation changed in the 15^{th}-16^{th} c. with the arrival of hoists equipped with counterweights. The practical value of these devices consisted in that they allowed a previously balanced load to be lifted by a tiny effort ("infinitesimally" small with respect to the load). In addition, the use of such devices permitted to reduce the size of the hoists which allowed them to be used within the narrow building sites of towns and in harbors.

The theoretical value of these machines — and this is what we are concerned about in this paper — consisted in that in their use both obstacles on the way of mathematization of motion have been eliminated: (i) the heterogeneity of forces involved and (ii) the alleged superiority of the joint force over the sum of its parts. In hoists with counterweight both forces — driving and resisting (the counterweight) — were of the same genus, so that the necessary condition for their comparison in terms of numerical ratio has been now fulfilled. Since gravity (or rather, weight) is additive, the equivalence of the force and the sum of its parts, has been ascertained as well. In addition, the use of the hoists with counterweights suggested the idea of reduction of their motion to the static condition of equilibrium, the solution of which — in terms of inverse proportionality of forces and arms — has been known from the times of Archimedes.

Acknowledgments

This work was supported by the Russian Foundation for Basic Research (project No. 15-03-00218a).

References

1. *Aristotle.* Mechanica // The Works of Aristotle. Vol. 6. / W.D. Ross (ed.). — London, Clarendon Press, 1913.
2. *Héron d'Alexandrie.* Les méchaniques ou l'élévateur / Carra de Vaux (trad. et publ.). — Paris, Imprimerie nationale, 1894.
3. *Galileo Galilei.* Mechanics // Galileo Galilei. Selected Works. Vol. 2. — M.: Nauka, 1964. — P. 5–38. (In Russian).
4. *Zaytsev E.A.* The Origins of Theoretical Mechanics (Antiquity, Middle Ages and the Beginning of the Modern Times) // S.I. Vavilov Institute for the History of Science and Technology. Annual Scientific Conference 2015. Vol. 1. — M.: Lenand, 2015. — P. 132–141. (In Russian).