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Probabilistic algorithms of constructing numerical solution of the Cauchy problem for systems of nonlinear parabolic equations

Ya. I. Belopolskaya[1], A.O. Stepanova[1]

[1] *Department of Mathematics,*
St Petersburg University of Architecture and Civil Engineering,
2-ja Krasnoarmejskaja str. 4 and POMI RAN, nab. Fontanka 27,
St. Petersburg, Russia

Abstract. In this paper we have two main goals. One of them is to construct stochastic processes associated with a class of systems of semilinear parabolic equations which allow to obtain a probabilistic representation of a solution to the Cauchy problem for a system from this class. The second goal is to reduce the Cauchy problem solution to a closed system of stochastic relations, prove the existence and uniqueness theorem for this system and apply it to develop a numerical algorithm to get a required solution of the problem under consideration.

Keywords: semilinear parabolic equations, Cauchy problem, diffusion processes, Markov chains, probabilistic representations, numerical solutions.

1. Introduction

Systems of the second order parabolic equations arise as mathematical models of various phenomena in physics, chemistry, biology, economics and finance and other fields. In some cases one can find that there exist stochastic processes associated with them (actually, not merely diffusion processes). This allows to interpret these systems as systems of forward or backward Kolmogorov equations for these processes. This interpretation not only reveals intrinsic links between macro and micro processes but also allows to develop new effective algorithms of numerical solution of the considered parabolic system.

In this paper we consider the backward Cauchy problem for a system of semilinear and quasilinear parabolic equations of the form

$$\frac{\partial u^m}{\partial s} + \mathcal{L}^u_m u^m + [Q^u u]^m = 0, \quad u^m(T,x) = u_0^m(x) \quad m = 1, \dots, d_1, \quad (1)$$

with a bounded smooth "initial" function $u_0(x) \in R^{d_1}, x \in R^d$. Here d, d_1 are fixed integers, $\mathcal{L}^v_m u^m = \langle a^m, \nabla u^m \rangle + \frac{1}{2} Tr A^m \nabla^2 u^m [A^m]^*$ and the matrix $Q^u = (q_{ml}(x,u))$ possesses the properties of a Q-matrix :

1) $q_{ml}(x,u)$ is uniformly bounded in $x \in R^d$ and has polylinear growth in $u \in R^{d_1}$ for fixed $l, m \in V = \{1, 2, \dots, d_1\}$;

2) $q_{ml}(x,u) \geq 0$ for any $x \in R^d, u \in R^{d_1}$ and $l \neq m$;

3) $q_{mm}(x,u) = -\sum_{l\neq m} q_{ml}(x,u)$ for any $x\in R^d, u\in R^{d_1}, m\in V$.

We aim first to construct stochastic processes which allow to obtain a probabilistic representation of a classical solution of the Cauchy problem (1) and apply this representation to construct a numerical approximation of the solution. Then we construct stochastic processes associated with a viscosity solution of the Cauchy problem (1) construct a numerical approximation of the solution.

2. Probabilistic representation of a classical solution of the Cauchy problem for a system of semilinear PDEs

Let $(\Omega,\mathcal{F},P)$ be a given probability space, $w(t)\in R^d$ be a Wiener process and $\gamma(t)$ be a Markov chain with the generator Q, defined on this space. Consider a system of relations including a stochastic differential equation (SDE) with respect to a process $\xi(t)\in R^d$

$$d\xi(t) = a^u(\xi(t),\gamma(t))dt + A^u(\xi(t),\gamma(t))dw(t), \xi(s)=x, \gamma(s)=m, \quad (2)$$

where $\gamma(t)$ is the Markov chain with the transition probability

$$P\{\gamma(t+\Delta t) = l|\gamma(t)=m, \xi(\theta), \gamma(\theta), \theta\leq t\} = q^u_{ml}(\xi(t))\Delta t + o(\Delta t), \quad (3)$$

when $l\neq m$ and a relation

$$u(s,x) = E[u_0(\xi_{s,x}(T), \gamma_{s,m}(T))]. \quad (4)$$

Here $a: R^d\times V\times R^{d_1}\to R^d, A: R^d\times V\times R^{d_1}1\to R^d\otimes R^d$ and we use notations of the form $a^u(x,m)\equiv a(x,m,u(x,m))$.

To construct a probabilistic representation of the classical solution to the Cauchy problem (1) we need an alternative representation of the Markov chain $\gamma(t)$ via stochastic integral with respect to a random Poisson measures $p(dz,dt)$.

To describe this representation for fixed $x\in R^d, v\in R^{d_1}$ we consider a random Poisson measure $p(dt,dz)$ defined on the space $\Omega\times[0,T]\times G$, where $G\subset R_+$. Next we choose a set of the consecutive left-closed, right-open intervals $\Delta_{lm}(x,v)$ covering G and having the length $q_{lm}(x,v) = q^v_{lm}(x)$ and let $Ep(dt,dz) = dtdz$, where dz is the Lebesgue measure.

At the end define a scalar function $g^{x,v}(m,z)$ by relations $g^{x,v}(m,z) = m-l$, when $z\in\Delta_{lm}(x,v)$, and 0 otherwise or $g^{x,v}(m,z) = \sum_{m=1}^M (m-l)I_{\{z\in\Delta_{lm}(x,v)\}}$. Then a stochastic differential of the Markov process $\gamma(t)$ with the transition probability (3) has the form

$$d\gamma(t) = \int_G g^u(\xi(t),\gamma(t-),z)p(dt,dz), \quad \gamma(s)=m,. \quad (5)$$

It should be mentioned that relations (2), (4), (5) make a closed system. To find a solution to this system we need some conditions on its coefficients.

Condition **C 1.** Coefficients $a^u(x,m), A^u(x,m)$ of SDE (2) are Lipschitz continuous in x and u, have a sublinear growth in x and polylinear growth in u for fixed m as well as the matrix $Q^u(x)$ is.

Theorem 1. Assume that for each $m \in V$ there exists a unique classical solution $u(s,x,m)$ of the Cauchy problem (1) and for this u_m coefficients of SDE (2) and (5) and the matrix $Q^u(X)$ satisfy **C 1.** Then there exists a unique solution of the system (2), (5) and the function $u(s,x,m)$ admits the representation (4).

Let us state conditions to ensure the existence and uniqueness of a solution of the system (2), (4),(5) and the function $u(s,x,m)$ given by (5) is a unique classical solution of the Cauchy problem (1).

We say that a triple $(\xi_{s,x}(t),\ \gamma_{s,m}(t),\ u(s,x,m))$ satisfies the system (2), (4), (5) if $(\xi_{s,x}(t), \gamma_{s,m}(t))$ is a two-component $\mathcal{F}_t$-adapted process such that

$$\xi(t) = x + \int_s^t a^u(\xi(\theta),\gamma(\theta))dt + \int_s^t A^u(\xi(\theta),\gamma(\theta))dw(\theta),$$
$$\gamma(t) = m + \int_s^t \int_R g^u(\xi(\theta),\gamma(\theta),z)p(d\theta,dz),$$

hold with probability 1 and $u(s,x,m)$ given by (3) are bounded Lipschitz continuous in z functions defined on $[0,T] \times R^d \times V$.

Condition **C 2**. Assume that **C 1** holds and $a(x,v,m), A(x,v,m), Q(x,v)$ are k-times differentiable in x and v, while $u_{0,m}(x)$ are bounded and k - times differentiable for each $m \in V$.

Theorem 2. *Assume that* **C 2** *holds for* $k=1$. *Then there exists an interval* $[T_1,T] \subset [0,T]$ *such that for all* $s \in [T_1,T]$ *there exists a unique solution of the system* (2),(4), (5).

Theorem 3. *Assume that* **C 2** *holds for* $k=2$. *Then there exists an interval* $[T_1,T] \subset [0,T]$, *Then there exists an interval such that for all* $s \in [T_2,T]$, $T_2 \le T_1$, *there exists a unique solution of the system* (2),(4), (5) *and* $u(s,x)$ *given by* (3) *is the unique classical solution of* (1).

To prove the existence of a solution to (2), (4), (5) we construct successive approximations

$$\xi^0(\theta) = x, \gamma^0(\theta) = m, u_1(\theta,x,m) = u_0^m(x),$$

$$\xi^1(t) = x + \int_s^t a(\xi^1(\theta),\gamma^1(\theta),u_1(\theta,\xi^1(\theta),\gamma^1(\theta))))d\theta,$$
$$+ \int_s^t A(\xi^1(\theta),\gamma^1(\theta),u_1(\theta,\xi^1(\theta),\gamma^1(\theta))))dw(\theta),$$

$$\gamma^1(t) = m + \int_s^t \int_R g(\xi_1(\theta), u_1(\theta, \xi^1(\theta)), \gamma^1(\theta), z) p(d\theta, dz),$$
$$u_2^m(s,x) = E[u_0^m(\xi^1_{s,x}(T), \gamma^1_{s,m}(T))].$$

We continue the recursive procedure to derive equations for processes $\xi^n_{s,x}(t), \gamma^n_{s,m}(t)$ and functions $u_n^m(s,x)$. We can prove under the above conditions [2] that $\xi^n_{s,x}(t), \gamma^n_{s,m}(t)$ converge in the square mean sense to solutions $\xi_{s,x}(t)$, and $\gamma_{s,m}(t)$ of SDE (3) and (5), while functions $u_n^m(s,x)$ for each $s \in (T_2, t]$ uniformly in x converge to a bounded Lipschitz continuous function $u^m(s,x)$.

Linear systems of the form (1) with coefficients $A^u(x) = A(x), a^u(x) = a(x), q^u(x) = q$ were studied in [1] , where probabilistic representations of a classical solution to the corresponding linear system were derived. These results were extended to nonlinear systems in [2] and this allows to use the representation (3) of the classical solution of the Cauchy problem (1) to obtain numerical results.

Let $s = t_0 \le t_1 \le \cdots \le t_n = T$, $\Delta t = t_{k+1} - t_k = \frac{T-s}{n}$, $\zeta \in \{-1, 1\}$ $P(\zeta = -1) = P(\zeta = 1) = \frac{1}{2}$. Applying the explicit Euler scheme, iteration process and the simplest approximation of a random variable $w(t+\Delta t) - w(t)$ by a random variable $\kappa = \zeta\sqrt{\Delta t}$, we develop an explicit method of solution to (1)

$$\bar{u}_1(\theta, x, m) = u_0(x, m), \bar{\xi}^0(\theta) = x, \bar{\gamma}^0(\theta) = m, \quad \theta \in [s, T],$$
$$\bar{\xi}^1(t_{k+1}) = x + a(x, m, \bar{u}_1(t_{k+1}, x, m))\Delta t$$
$$+A(x, m, \bar{u}_1(t_{k+1}, x, m))\zeta\sqrt{\Delta t},$$
$$\bar{\gamma}(t_{k+1}) = l + \int_R g(m, z) p(\bar{u}^1(t_{k+1}, x, m), \Delta t, dz),$$
$$\bar{u}(t_k, x, m) = E\bar{u}(t_{k+1}, \bar{\xi}_{t_k,x}(t_{k+1}), \bar{\gamma}_{t_k,l}(t_{k+1}))$$
$$= \frac{1}{2}\sum_{j=1}^M [\bar{u}(t_{k+1}, x + \tilde{a}_k\Delta t + \tilde{A}_k\sqrt{\Delta}t, j) + \bar{u}(t_{k+1}, x + \tilde{a}_k\Delta t - \tilde{A}_k\sqrt{\Delta}t, j] q^{\bar{u}}_{mj}\Delta t,$$

where $\tilde{a}_k = a(\xi(t_k), \bar{u}(t_{k+1}, \xi(t_k), \gamma(t_k)), \tilde{A}_k = A(\xi(t_k), \bar{u}(t_{k+1}, \xi(t_k), \gamma(t_k))$. Since the two component process $(\xi(\theta), \gamma(\theta)) \in R^d \times V$ is a Markov process, we know that the relation

$$u(s, x, m) = Eu_0(\xi_{s,x}(T), \gamma_{s,m}(T)) = U(s, T)u_0(x, m) \tag{6}$$

gives rise to a nonlinear evolution family $U(s, T)$. Set

$$\bar{U}(t_k, t_{k+1}u = \bar{u}(t_k, x, m) = Eu(t_{k+1}, \bar{\xi}_{t_k,x}(t_{k+1}), \bar{\gamma}_{t_k,m}(t_{k+1}))$$

and note that similar to the linear case [3], under **C 2** condition we get an estimate

$$\sup_x |u_m(t_k,x) - \bar{u}_m(t_k,x)| \le C(\Delta t)^2.$$

This results (see [4]) that one can approximate the evolution family $U(s,t)$ defined by (6) using a (nonevolution) family

$$V_n(s,T)u_0(x,m) = \prod_{k=1}^{n} \bar{U}(t_{k-1},t_k)u_0(x,m).$$

3. Conclusions

We have derived probabilistic representations of classical solutions of the Cauchy problem for a class of semilinear parabolic equations. These representations are used to construct an explicit algorithm for obtaining a numerical solution to the considered Cauchy problem.

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