UDC 517.521

On a generalization of the Leibniz theorem

G. A. Zverkina[*†]

[*] *Moscow State University of Railway Engineering (MIIT)*
9b9 Obrazcova Street, Moscow, 127994, Russia

[†] *V. A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences*
65 Profsoyuznaya street, Moscow, 117997, Russia

Abstract. The well-known Leibniz theorem (Leibniz Criterion or alternating series test) of convergence of alternating series is generalised for the case when the absolute value of terms of series are "not absolutely monotonously" convergent to zero. Questions of accuracy of the estimation for the series remainder are considered.

Keywords: alternating series test, Leibniz theorem, convergence rate, Leibniz Criterion.

1. Introduction

The Leibniz theorem (Leibniz Criterion or alternating series test) provides the possibility to demonstrate the convergence of an alternating series with decreasing to zero components. However in some cases values of series components decrease to zero fluctuating. In some this case it can use the facts proved below.

2. Generalization of the Leibniz theorem

Definition 1. *The sequence $\{a_n\}$ is called $Z(w)$-monotonously increasing (decreasing) on set $\mathfrak{D}$ $(w \in \mathbb{N})$ if $\forall k \in \mathfrak{D}$ it is carried out $a_{k+w} \geqslant a_k$ (respectively $a_{k+w} \leqslant a_k$).* $\Diamond$

Theorem 1. *If the sequence a_n is $Z(2w-1)$-monotonously decreasing for $n \geqslant n_0$ $(w, n \in \mathbb{N})$ and $\lim\limits_{n\to+\infty} a_n = 0$, then the series $\sum\limits_{n=n_0}^{\infty}(-1)^n a_n$ converges. And $|R_m| \leqslant \sum\limits_{n=m+1}^{m+2w} a_k$, where R_m is the series remainder: $R_m \stackrel{\text{def}}{=} \sum\limits_{n=m+1}^{\infty}(-1)^n a_n$.* $\square$

Remark 1. $Z(2w)$-monotonicity of $\{a_n\}$ and $\lim\limits_{n\to+\infty} a_n = 0$, not imply the convergence of $\sum\limits_{k=n_0}^{\infty}(-1)^n a_n$. $\Diamond$

Example 1. $a_n = \begin{cases} k^{-2} & \text{if} \quad n = 2k, \\ k^{-1} & \text{if} \quad n = 2k-1, \end{cases} \qquad \sum\limits_{n=1}^{\infty}(-1)^n a_n = -\infty.$ $\square$

If $w=1$, then $Z(2w-1)$-monotonicity turns into usual monotonicity ($0 \leqslant a_{n+1} \leqslant a_n$), and the Theorem 1 turns into the well-known Leibniz Theorem about alternating series:

Theorem 2 (G.W. von Leibniz, 1682). *If $a_n \searrow 0$, then the series $S=\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.* □

Corollary 1. Denote $R_n=\sum\limits_{i=n+1}^{\infty}(-1)^{i+1}a_i$. If $a_n \searrow 0$, then

$$|R_m| \leqslant R_m^L \stackrel{\text{def}}{=} a_{m+1}; \tag{1}$$

$$\text{or } R_m=\theta \cdot a_{m+1}, \qquad 0 \leqslant \theta \leqslant 1. \tag{2}$$

□

Denote *Z-series* – a series satisfying to conditions of Theorem 1, and *L-series* – a series satisfying to conditions of Theorem 2. It's well-known, the Leibniz theorem is a special case of Dirichlet's theorem (Dirichlet test):

Theorem 3 (J.P.G. Lejeune Dirichlet). *If $\forall N \in \mathbb{N}$ $\left|\sum\limits_{n=1}^{N} b_n\right| < M$, where M is some constant, and for all $n \in \mathbb{N}$, $a_n \geqslant a_{n+1}$, $\lim\limits_{n\to\infty} a_n = 0$, then the series $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.* □

Below (Examples 2, 3, 4) we give samples of series for which Theorem 1 allows to prove convergence, but Dirichlet's Theorem is inapplicable or its application involves the big technical difficulties.

Proof of Theorem 1 represents the $Z(2w-1)$-series as the sum of $(2w-1)$ convergent series; we skip it here (see [1]). ■

Remark 2. $S-S_m=\sum\limits_{j=1}^{2w-1}(-1)^{m+j}\tilde{\theta}_{m+j}a_{m+j}$, $0 \leqslant \tilde{\theta}_r \leqslant 1$, or

$|S-S_m| \leqslant R_m^Z \stackrel{\text{def}}{=} \max\left\{\sum\limits_{r=1}^{w-1} a_{m+2r}, \sum\limits_{r=1}^{w} a_{m+2r-1}\right\}$. Generally, this estimation is not improved asymptotically. ♢

Example 2. Let the sequence a_n be defined as follows:

$$a_n=\begin{cases} \frac{1}{k}+\frac{1}{2^k}, & \text{if} \quad n=3(2k-1)-2 \text{ or } n=3(2k-1); \\ \frac{1}{10^k}, & \text{if} \quad n=3(2k-1)-1 \text{ or } n=3\cdot 2k-1; \\ \frac{1}{k}, & \text{if} \quad n=3\cdot 2k-2 \text{ or } n=3\cdot 2k; \end{cases} \quad k \in \mathbb{N}.$$

$\lim\limits_{n\to\infty} a_n = 0$, $\{a_n\}$ is $Z(3)$-monotonous, and a series $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ is not L-series; $S=\sum\limits_{n=1}^{\infty}(-1)^n a_n = 2$. Here $R_{6k+3} \sim \frac{2}{k}$, $R_{6k}=\frac{1}{2^{k-1}}$, that is the absolute value of the remainder of a series has big fluctuations. This series cannot be easily examined by Dirichlet test (Theorem 3). □

3. Conditions of applicability of the Theorem 1

Often components of a numerical series represent values of some continuous function in integer points: $a_n = f(n)$. Therefore for research of convergence of series $\sum\limits_{n=1}^{\infty}(-1)^n f(n)$ in the case when $f(x)$ is not a monotonous function, it is natural to extend the concept of Z-monotonicity to continuous functions.

Definition 2. Function $f(x)$ is called $Z(T)$-monotonously increasing (decreasing) on the set $\mathfrak{D}$ $(T > 0)$ if for all $x \in \mathfrak{D}$ it is carried out $f(x+T) \geqslant f(x)$ (respectively $f(x+T) \leqslant f(x)$). ♢

But the fact, that $f(x)$ is $Z(T)$-monotonous function, not implies that the sequence $f(n)$ is $Z(k)$-monotonous. E.g., the function $\varphi(x) = \ln x + x\sin^2 x$ is $Z(2\pi)$-monotonously increasing for $x > 0$, however for any natural k it is not $Z(k)$-monotonous. Therefore it is necessary to introduce the concept of the strong (or very) Z-monotony.

Definition 3. *Function $f(x)$ is called Zv-monotonously*[1] *increasing (decreasing) on set $\mathfrak{D}$, if there exists $T > 0$ such that $f(x+T+\tau) \geqslant f(x)$ (respectively $f(x+T+\tau) \leqslant f(x)$) for all $x \in \mathfrak{D}$, and for all $\tau > 0$.* ♢

Let us introduce the parameter of Zv-monotonous increasing [decreasing] function on set $\mathfrak{D}$: $Par_{Zv}(f(x)) \stackrel{\text{def}}{=} \inf\{T > 0 : \forall\tau > 0, \quad \forall x \in \mathfrak{D},\ f(x+T+\tau) \geqslant [\leqslant] f(x)\}$. If $Par_{Zv}(f(x)) = 0$, then $f(x)$ is monotonous in usual sense.

If $f(x)$ is Zv-monotonous functions $f(x)$ on a set $\mathfrak{D}$, then there exists the monotonous[2] function $\varphi(x)$, such that $\forall t \in \mathfrak{D}$ the value $f(t)$ is between numbers $\varphi(t)$ and $\varphi(t+T)$, where $T \geqslant Par_{Zv}(f(x))$. Therefore for the proof of Zv-monotonous increase (decrease) of function $f(x)$ it is enough to find two monotonous increasing (or decreasing) functions $\varphi_1(x)$, $\varphi_2(x)$, such that $\varphi_1(x) \leqslant f(x) \leqslant \varphi_2(x)$ and there exists $T > 0$ such that $\varphi_1(x+T) > \varphi_2(x)$ for all $x \in \mathfrak{D}$ (here $T \geqslant Par_{Zv}(f(x))$. In most cases it is difficult to define parameter Zv-monotonous function, but it is possible to obtain an estimation of this parameter.

Example 3. Let $0 < \alpha \leqslant 1$ and $p(x)$ is bounded: $|p(x)| < M$. Then $f(x) = x^\alpha + p(x)x^{\alpha-1}$ is Zv-monotonously increasing for $x > \frac{(1-\alpha)M}{\alpha}$. □

Theorem 4. *If $f(x)$ Zv-monotonously decreases and $\lim\limits_{x\to\infty} f(x) = 0$ then a series $\sum\limits_{n=n_0}^{\infty}(-1)^n f(n)$ converges.* □

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[1] *Z-very*-monotonously.

[2] Non-strict monotonicity means: $\varphi(x)$ is monotonous on $\mathfrak{D}$, if $\forall a < b \in \mathfrak{D}, \quad \varphi(a) \leqslant \varphi(b)$; or $\forall a < b \in \mathfrak{D} \quad \varphi(a) \geqslant \varphi(b)$.

Proof. Let us find some odd number $2w-1 \geqslant Par_{Zv}(f(x))$. Then $\{f(n)\}$ is $Z(2w-1)$-monotonous, and $\sum\limits_{n=n_0}^{\infty}(-1)^n f(n)$ converges. ■

Example 4. The series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n n^\beta}{n+p(n)}$ converges, if $0\leqslant \beta < 1$ and the function $p(x)$ is bounded (see Example 3). □

Example 5. The function $g(x) = \frac{1}{x+2\cos x}$ is Zv-monotonous and $Par_{Zv}(g(x)) \leqslant 2\pi < 7$. Hence, $\sum\limits_{n=1}^{\infty}\frac{(-1)^{(n-1)}}{n+2\cos n}$ is convergent Z-series. So, the series remainder $R_m = \sum\limits_{n=m+1}^{\infty}\frac{(-1)^{(n-1)}}{n+2\cos n} \leqslant a_{m+1}+a_{m+3}+a_{m+5}+a_{m+7}$. This estimate and bounds for $Par_{Zv}(g(x))$ can be improved. □

4. On the accuracy of the estimation of the remainder of L-series and Z-series.

For a long time it is noticed, that the estimation (2) in most cases gives very good accuracy. But, as the L-series which research differently as by means of a criterion (theorem) of Leibniz is impossible, its converge usually very slowly, and it would be desirable to have a method of specification of estimations from Corollary 1 or Remark 2.

Example 6. For a well-known series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln 2$ the estimation (1) gives an inequality $|R_m| = \left|\sum\limits_{n=m+1}^{\infty}\frac{(-1)^{n+1}}{n}\right| \leqslant R_m^L = \frac{1}{m+1}$. More accurately: $|R_m| = \frac{1}{m+1} - \frac{1}{m+2} + \frac{1}{m+3} - \frac{1}{m+4} + \ldots = \sum\limits_{k=1}^{\infty}\frac{1}{(m+2k-1)(m+2k)}$; $\frac{1}{2}\ln\left(\frac{m+2}{m+1}\right) < |R_m| < \frac{1}{2}\ln\left(\frac{m}{m-1}\right)$; $|R_m| \sim \frac{1}{2m}$ as $m \to \infty$. So, the relative error of (1) is about $\frac{1}{2}$. □

Example 7. Let's consider another L-series: $a_n = \begin{cases} \frac{1}{k}, & \text{if } n = 2k-1; \\ \frac{1}{k} - \frac{1}{2^k}, & \text{if } n = 2k, \end{cases}$

or $a_n = \frac{1}{\left[\frac{n+1}{2}\right]} - \frac{1+(-1)^n}{2^{\frac{n}{2}+1}}$. For $n > 7$, $a_n \downarrow 0$, and $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n = 1$.

Further, $R_n = \begin{cases} \frac{1}{2^k} - \frac{1}{k}, & \text{if } n = 2k-1; \\ \frac{1}{2^k}, & \text{if } n = 2k; \end{cases}$ $|R_{2n-1}| \sim \frac{2}{n+1}$, $|R_{2n}| \sim \frac{1}{2^n}$, and accuracy of an estimation has a big fluctuations. In this case it is possible to speak about unsatisfactory accuracy of an estimation (1). The presented case has some similarity to Example 2. □

Theorem 5. *If the sequence* $\{a_n\}$ *monotonously decreases to* 0 $(a_n \downarrow 0)$ *at* $n > n_0$, *and at* $n > n_0$ *the condition* $a_{n+1} \leqslant \frac{a_n + a_{n+2}}{2}$ *is satisfied, then for* $m > n_0$, $\frac{1}{2}a_{m+1} \leqslant |R_m| \leqslant \frac{1}{2}a_m$. □

We skip the proof (see [1]).

Corollary 2. *For a* $Z(p)$*-series* ($p = 2w - 1$) $\sum\limits_{n=n_0}^{\infty} (-1)^n a_n$ *it is possible to give the following general estimation of the remainder of a series:* $R_m = (-1)^m(\delta_1 - \delta_2 + \delta_3 - \ldots - \delta_{p-1} + \delta_p)$, *where* δ_k *is a remainder of* L*-series. Thus,* $\frac{1}{2}a_{n+i} \leqslant \delta_i \leqslant a_{n+i}$. *Therefore*

$$|R_m| \leqslant a_{m+1} - \frac{1}{2}a_{m+2} + a_{m+3} - \frac{1}{2}a_{m+4} + \ldots - \frac{1}{2}a_{m+p-1} + a_{m+p} \leqslant$$
$$\leqslant \max_{s=0,\ldots,w}\{a_{m+2s-1}\} \cdot \frac{p+1}{2} - \frac{1}{2} \min_{s=1,\ldots,w-1}\{a_{m+2s}\} \cdot \frac{p-1}{2}.$$ □

In Example 5 we have: a series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n+2\cos n}$ converges, and $|R_m| \lesssim \frac{1}{2} \cdot \frac{1}{m}$.

5. Some remarks

Theorem 1 can be generalized by different natural way. For example, let denote the series $\sum\limits_{k=0}^{\infty} a_k$ ($a_k \neq 0$ for all $k \in \mathbb{N}$) as a w-periodical-single series if for some $w \in \mathbb{N}$ it is carried out for all $k > k_0$ sign $(a_k) = -$sign (a_{k+w}), here sign $(a_k) = 1$ if $a_k > 0$ and sign $(a_k) = -1$ if $a_k > 0$. Then if consequence $\{|a_k|\}$ is a $Z(w)$-monotonously decreasing to zero, then the series $\sum\limits_{k=0}^{\infty} a_k$ converges. The estimates for the remainder of this series can be found by the reasonings above. This example include the situation when some subseries (L-series) of Z-series are zero. In other words, the w-periodical-single series can be transformed to the $Z(2n-1)$-series by addition of some quantity of zero-series.

Acknowledgments

The work is partially supported by RFBR grant No 17-01-00633 A. Author expresses gratitude to Professor V.N.Chubarikov (Department of mechanics and mathematics of Lomonosov Moscow State University).

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