UDC 519.248, 517.929

# Analytic-Numerical Investigation of the Dual Risk Model with Investments: Survival Probability Functions as the Solutions of Singular Problems for Integro-Differential Equations

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**Abstract.** We study the life annuity insurance model when simple investment strategies (SISs) of the two types are used: risky investments and risk-free ones. According to a SIS of the first type, the insurance company invests a constant positive part of its surplus into a risky asset while the remaining part is invested into a risk-free asset. A risk-free SIS means that the whole surplus is invested into a risk-free asset. We formulate and study some associated singular problems for linear integro-differential equations (IDEs). For the case of exponential distribution of revenue sizes, we state that survival probabilities as the functions of the initial surplus (IS) are unique solutions of the corresponding problems. Using the results of computational experiments, we conclude that in the region of small sizes of IS the risky SIS may be more effective tool for increasing of the survival probability than risk-free one.

**Keywords:** survival probability, dual risk model, risky and risk-free investments, integro-differential equations, singular problems.

## 1. Introduction

We consider the life annuity insurance model [1], where the surplus of a company (in the absence of investments) is of the form

$$R_t = u - ct + \sum_{k=1}^{N(t)} Z_k, \quad t \geq 0. \tag{1}$$

Here $R_t$ is the surplus of a company at time $t \geq 0$; $u$ is the IS, $c > 0$ is the life annuity rate (or the pension payments per unit of time), assumed to be deterministic and fixed. $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$ that, for any $t > 0$, determines the number of random revenues up to the time $t$; $Z_k$ $(k = 1, 2, ...)$ are independent identically distributed random variables with a distribution function $F(z)$ $(F(0) = 0$, $\mathbf{E}Z_1 = m < \infty)$ that determine the revenue sizes and are assumed to be independent of $N(t)$.

The considered insurance model is dual to the classical non-life collective risk model (well-known as the Cramér-Lundberg model [1]). In comparison with the classical model, the circumstances in the dual model are reversed: the components of the insurance risk process obtain opposite signs, so that jumps in the dual model are positive, while the deterministic component becomes decreasing (due to pension payments). The jumps of the process determine the revenue sizes; these revenues arise at the moments of the death of policyholders. The considered model is also called "dual risk model" [2].

Let a fixed part $\alpha$ of the surplus be continuously invested into risky asset with price $S_t$ following the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dw_t, \quad t \geq 0,$$

where $\mu$ is the stock return rate, $\sigma$ is the volatility, $w_t$ is a standard Brownian motion independent of $N(t)$ and $Z_i$'s. The rest part $(1-\alpha)$ of the surplus is invested in a risk-free asset which evolves as $dB_t = rB_t\, dt$, $t \geq 0$, where $r > 0$ is the interest rate.

Then the resulting surplus process $X_t$ is governed by the equation

$$dX_t = \alpha(\mu - r)X_t dt + rX_t dt + \alpha\sigma X_t dw_t + dR_t, \quad t \geq 0, \tag{2}$$

with the initial condition $X_0 = u$, where $R_t$ is defined by (1).

If $0 < \alpha \leq 1$ then there is a risky SIS, and if $\alpha = 0$ then we have the risk-free SIS.

**Remark 1** *The case $\alpha > 0$ is equivalent to the case when whole of the surplus is invested into risky asset with modified parameters $\mu_\alpha = \alpha\mu + (1-\alpha)r$, $\sigma_\alpha = \alpha\sigma$. Then, for the corresponding surplus process, the equation* (1) *is fulfilled with $\alpha = 1$ and $\mu_\alpha$, $\sigma_\alpha$ instead of $\mu$ and $\sigma$ respectively:*

$$dX_t = \mu_\alpha\, X_t\, dt + \sigma_\alpha\, X_t\, dw_t + dR_t, \quad t \geq 0. \tag{3}$$

*For the case $\alpha = 0$, we define $\mu_0 = r$, $\sigma_0 = 0$, and the equations* (1) *and* (1) *have the same form:*

$$dX_t = r\, X_t\, dt + dR_t, \quad t \geq 0. \tag{4}$$

Denote by $\varphi(u)$ the survival probability (SP): $\varphi(u) = \mathbf{P}\,(X_t \geq 0,\ t \geq 0)$. Then $\Psi(u) = 1 - \varphi(u)$ is the ruin probability (RP).

Let $\varphi_0(u) = \mathbf{P}\,(R_t \geq 0,\ t \geq 0)$ be the SP for the process (1) and $\Psi_0(u) = 1 - \varphi_0(u)$ be the corresponding RP (in the absence of investments). In the case of exponential distribution of revenue sizes, namely when

$$F(z) = 1 - \exp\left(-z/m\right), \quad m > 0, \tag{5}$$

and if the safety loading is positive, i.e., the inequality $\lambda m > c$ is valid, it is easy to show that $\Psi_0(u) = \exp\big(-(\lambda m - c)\, u/(mc)\big)$.

**Theorem 1** [3] *For the process* (1), *let* $F(z)$ *be defined by* (1), $m > 0$, $\sigma_\alpha^2 > 0$, $\mu_\alpha > 0$, $\beta := 2\mu_\alpha/\sigma_\alpha^2 - 1$. *Then:* (1) *if* $\beta > 0$ *then* $\Psi(u) = Ku^{-\beta}(1 + o(1))$, $u \to \infty$, *for some constant* $K > 0$; (2) *if* $\beta \leq 0$ *then* $\Psi(u) = 1$, *for any* $u \geq 0$.

This statement in combination with the exponential representation in corresponding model without investments leads to the conclusion that investment of the constant part of the surplus into risky assets can impair the insurers solvency at least in the region of large values of IS.

Main goal of our paper is to identify the impact of risky SISs on the solvency in comparison to the effect of risk-free SIS in the dual risk model for all possible IS values. For the case of risky investments, some results of given paper (along with other ones) are briefly represented in [5].

## 2. Main results

**2.1. Preliminary propositions.** For $\alpha \geq 0$, we will consider the process $X_t = X_t^\alpha$ defined by (1) with initial state $X_0 = u$. Recall at first that the infinitesimal generator $\mathcal{A}^\alpha$ of the process $X_t^\alpha$ has the form

$$(\mathcal{A}^\alpha f)(u) = \frac{1}{2}\sigma_\alpha^2 u^2 f''(u) + f'(u)(\mu_\alpha u - c) - \lambda f(u) + \lambda \int_0^\infty f(u+z)\, dF(z), \tag{6}$$

for any function $f(u)$ from a certain subclass of the space of real-valued, continuously differentiable (on some intervals) functions; more precisely, in the case $\alpha > 0$ we deal with $\mathcal{C}^2(\mathbb{R}_+)$ of twice continuously differentiable on $(0, \infty)$ functions.

For the case $\alpha > 0$, it is emphasized in [3] that the main difficulty in deriving the corresponding IDE

$$(\mathcal{A}^\alpha \varphi)(u) = 0, \quad u > 0, \tag{7}$$

is to prove the smoothness of the SP $\varphi(u) = \varphi_\alpha(u)$ of the process $X_t^\alpha$. In this paper we apply the approach based on sufficiency principle [4] which allows us to avoid the a priori proof of the smoothness of the SP on the corresponding interval as well as the justification of the boundary conditions at infinity (note that, for $\alpha = 0$, the SP is not smooth on $(0, \infty)$ in general case). We apply this approach to the case $\alpha > 0$ as well as to the case $\alpha = 0$.

**Definition 1** *Let* $\mathcal{K}$ *be the class of functions* $\varphi(u)$ *belonging to* $\mathcal{C}^2(\mathbb{R}_+)$ *and satisfying conditions*

$$\lim_{u \to +0} \varphi(u) = 0, \qquad \lim_{u \to +\infty} \varphi(u) = 1. \tag{8}$$

*Denote also by* $\mathcal{L}$ *the class of functions* $\varphi(u)$ *defined on* $[0,\infty)$, *continuously differentiable on* $(0, c/r)$ *and satisfying conditions*

$$\varphi(0)=0, \qquad \lim_{u\to c/r-0} \varphi(u)=1, \qquad \varphi(u)=1, \ \ u \geq c/r. \tag{9}$$

For the proof of the following lemmas, we use the approach of [4].

**Lemma 1** *Let all the parameters in* (2) *be positive numbers and the inequality*

$$2\mu_\alpha > \sigma_\alpha^2 \tag{10}$$

*be fulfilled. Suppose IDE* (2) *has a solution* $\varphi \in \mathcal{K}$. *Then, for arbitrary* $u \geq 0$, $\varphi(u)$ *is the SP for the process* (1) *with initial state* $X_0 = u$.[1]

**Lemma 2** *For* $\alpha = 0$, *let in* (2) *all the parameters be fixed numbers, where* $c > 0$, $\lambda > 0$, $\mu_0 = r > 0$, $\sigma_0 = 0$. *Let the function* $\varphi \in \mathcal{L}$ *be satisfying IDE* (2) *for all* $u > 0$ *(perhaps, with exception of the point* $c/r$*). Then, for arbitrary* $u \geq 0$, $\varphi(u)$ *is the SP for the process* (1) *with initial state* $X_0 = u$.

**2.2. Main results for the case of exponential distribution of revenue sizes.** Using the results formulated above and some preliminary investigations of the corresponding singular problems for IDE (2) with $F(z)$ of the form (1), we establish the following statements for the process defined by (1), where $\alpha \geq 0$.

**Theorem 2** *For* $\alpha > 0$, *let* $\mu_\alpha > 0$, $\sigma_\alpha \neq 0$ *and* (2) *be satisfied. Then:* (I) *the SP* $\varphi(u)$ *of the process* (1) *with* $X_0 = u$ *belongs to the class* $\mathcal{K}$ *and is the unique solution in this class to the singular boundary value IDE problem* (2), (2); (II) $\varphi(u)$ *may be defined by the formula* $\varphi(u) = 1 - \int\limits_u^\infty \psi(s)ds$, *where* $\psi(u) = \varphi'(u)$ *is the solution on* $\mathbb{R}_+$ *of the singular problem for ODE:*

$$\frac{1}{2}\sigma_\alpha^2 u^2 \psi''(u) + (\mu_\alpha u + \sigma_\alpha^2 u - c - \frac{1}{2m}\sigma_\alpha^2 u^2)\psi'(u) +$$

$$+(\mu_\alpha - \lambda - \frac{\mu_\alpha u - c}{m})\psi(u) = 0, \qquad u > 0,$$

$$\lim_{u\to\infty} \psi(u) = \lim_{u\to\infty} \psi'(u) = 0, \qquad \int_0^\infty \psi(s)ds = 1.$$

**Theorem 3** *For* $\alpha = 0$, *let* $\mu_0 = r > 0$, $\sigma_0 = 0$. *Then:* (I) *the SP* $\varphi(u)$ *of the process* (1) *with* $X_0 = u$ *belongs to the class* $\mathcal{L}$ *and is the unique solution in this class to the singular IDE problem* (2), (2) *(it satisfies* (2)

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[1] For the SP in the case of risky investments, the formulation of Theorem 2 in [5] contains a mistake: the first condition from (2) is absent therein.

*in the sense of Lemma* 2*);* (II) $\varphi(u)$ *on the interval* $[0, c/r)$ *has the form* $\varphi(u) = 1 - \int\limits_u^{c/r} \psi(s)\,ds$*, where* $\psi(u)$*,* $u \in [0, c/r)$*, is defined by the formula*

$$\psi(u) = \left[ \int\limits_0^{c/r} (c/r-u)^{\lambda/r-1} \exp(u/m)\,du \right]^{-1} (c/r-u)^{\lambda/r-1} \exp(u/m).$$

Let us remark that, for $\alpha = 0$ and $\lambda \le r$, the SP is *non-smooth viscosity solution* of the IDE (2) (concept of the *viscosity solutions* is applied to analogous model in [6]).

## 3. Conclusions

The studies given in previous sections allow us to suggest computationally simple and theoretically justified algorithms for numerical calculation of the SP in the considered models with SISs. We use IDE approach and so called sufficiency principle based on verification arguments [4], which state that the solutions of certain singular problems for IDEs define the corresponding SPs. Computational experiments show that, for small values of IS, risky SIS with moderate volatility can be more effective way to minimize the probability of bankruptcy although risk-free investment provides the survival with probability 1 for IS greater than $c/r$.

## References


1. *Grandell J.* Aspects of Risk Theory. — Springer, 1991.
2. *Albrecher H., Badescu A., Landriault D.* On the dual risk model with tax payments // Insurance Math. Econom. — 2008. — Vol. 42. — P. 1086–1094.
3. *Kabanov Yu., Pergamenshchikov S.* In the insurance business risky investments are dangerous: the case of negative risk sums // Finance Stochast. — 2016. — Vol. 20, no. 2. — P. 355–379.
4. *Belkina T.* Risky investment for insurers and sufficiency theorems for the survival probability // Markov Processes Relat. Fields. — 2014. — Vol. 20. — P. 505–525.
5. *Belkina T.A., Konyukhova N.B., Slavko B.V.* Survival probability in the life annuity insurance model with stochastic return on investments // CEUR-WS — 2016. — Vol. 1726. — P. 1–12 (http://ceur-ws.org/Vol-1726/).
6. *Belkina T., Kabanov Yu.* Viscosity solutions of integro-differential equations for nonruin probabilities // Theory Probab. Appl. — 2016. — Vol. 60, no. 4. — P. 671–679.