UDC 519.21

INTEGRO-LOCAL LIMIT THEOREMS FOR MULTIDIMENSIONAL COMPOUND RENEWAL PROCESSES

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Abstract. We establish integro-local limit theorems for compound renewal processes.

Keywords: Compound renewal process, Cramer's condition, large deviations, integro-local limit theorem.

1. Formulation of the problem

Let $(\tau, \boldsymbol{\zeta}), (\tau_1, \boldsymbol{\zeta}_1), (\tau_2, \boldsymbol{\zeta}_2), \cdots$—be a sequence i.i.d. random vectors,

$$\tau > 0, \quad \boldsymbol{\zeta} = (\zeta_{(1)}, \cdots \zeta_{(d)}) \in \mathbb{R}^d, \quad d \geq 1.$$

Let

$$T_0 := 0, \quad T_{n+1} := T_n + \tau_{n+1}, \quad \mathbf{Z}_0 := \mathbf{0}, \quad \mathbf{Z}_{n+1} := \mathbf{Z}_n + \boldsymbol{\zeta}_{n+1};$$

let for $t \geq 0$

$$\eta(t) := \min\{k \geq 0 : \ T_k > t\}, \quad \nu(t) := \max\{k \geq 0 : \ T_k \leq t\}.$$

The compound renewal processes $\mathbf{Z}(t)$, $\mathbf{Y}(t)$ for the sequence $(\tau_j, \boldsymbol{\zeta}_j), \ j \geq 1$, are defined as (see [1], [2])

$$\mathbf{Z}(t) := \mathbf{Z}_{\nu(t)}, \qquad \mathbf{Y}(t) := \mathbf{Z}_{\eta(t)} = \mathbf{Z}(t) + \boldsymbol{\zeta}_{\eta(t)} \quad t \geq 0.$$

Let the Cramer's moment condition hold:
$[\mathbf{C}_0]$. For some $\delta > 0$

$$\mathbf{E}e^{\delta\tau + \delta|\boldsymbol{\zeta}|} < \infty.$$

For a vector $\mathbf{x} = (x_{(1)}, \cdots x_{(d)}) \in \mathbb{R}^d$ let

$$\Delta[\mathbf{x}) := \prod_{j=1}^{d} [x_{(d)}, \ x_{(d)} + \Delta), \quad \Delta = \Delta_T > 0,$$

$\Delta \to 0$ slowly enough in not lattice case and $\Delta = 1$ in arithmetic case. We study integro-local limit theorems for $\mathbf{Z}(T)$, $\mathbf{Y}(T)$ as $T \to \infty$, i.e. the exact asymptotics for the probabilities

$$\mathbf{P}(\mathbf{Z}(T) \in \Delta[\mathbf{x})) =?, \quad \mathbf{P}(\mathbf{Z}(T) \in \Delta[\mathbf{x})) =?$$

in the range of normal and large deviations.

This is joint work with E.I.Prokopenko.

2. Deviation (rate) function

For $(\lambda, \boldsymbol{\mu}) = (\lambda, \mu_{(1)}, \cdots \mu_{(d)}) \in \mathbb{R}^{d+1}$ let us define

$$A(\lambda, \boldsymbol{\mu}) := \mathbf{E} e^{\lambda\tau + \boldsymbol{\mu}\boldsymbol{\zeta}},$$

where $\boldsymbol{\mu}\boldsymbol{\zeta} := \mu_{(1)}\zeta_{(1)} + \cdots + \mu_{(d)}\zeta_{(d)}$;

$$\mathcal{A}^{\leq 0} := \{(\lambda, \boldsymbol{\mu}) : \ A(\lambda, \boldsymbol{\mu}) \leq 0\}, \qquad \lambda_+ := \sup\{\lambda : \ \mathbf{E} e^{\lambda\tau} < \infty\}.$$

Then for $\boldsymbol{\mu} \in \mathbb{R}^d$ put

$$A(\boldsymbol{\mu}) := -\sup\{\lambda : \ (\lambda, \boldsymbol{\mu}) \in \mathcal{A}^{\leq 0}\},$$

where $\sup\{\lambda : \ \lambda \in \emptyset\} = -\infty$;

$$\hat{A}(\boldsymbol{\mu}) := \max\{A(\boldsymbol{\mu}), \lambda_+\}.$$

Finely, for $\boldsymbol{\alpha} \in \mathbb{R}^d$ define deviation (rate) functions $D(\boldsymbol{\alpha})$ and $\hat{D}(\boldsymbol{\alpha})$:

$$D(\boldsymbol{\alpha}) := \sup_{\boldsymbol{\mu}}\{\boldsymbol{\mu}\boldsymbol{\alpha} - A(\boldsymbol{\mu})\}, \quad \hat{D}(\boldsymbol{\alpha}) := \sup_{\boldsymbol{\mu}}\{\boldsymbol{\mu}\boldsymbol{\alpha} - \hat{A}(\boldsymbol{\mu})\}.$$

3. The Large Deviation Principle

Definition 1. *Process $\{\frac{\mathbf{Z}(T)}{T}\}$ satisfies the large deviation principle (LDP) in $\mathbb{R}^d$ with a good rate function $\hat{D}(\boldsymbol{\alpha})$, if for all sets $B \subset \mathbb{R}^d$*

(i)

$$\limsup_{T\to\infty} \frac{1}{T} \ln \mathbf{P}(\frac{\mathbf{Z}(T)}{T} \in B) \leq -\inf_{\alpha\in[B]} \hat{D}(\boldsymbol{\alpha}),$$

where $[B]$ is closure of B;

(ii)

$$\liminf_{T\to\infty} \frac{1}{T} \ln \mathbf{P}(\frac{\mathbf{Z}(T)}{T} \in B) \geq -\inf_{\alpha\in(B)} \hat{D}(\boldsymbol{\alpha}),$$

where $[B]$ *is interior of* B*;*

(iii) the set $\{\boldsymbol{\alpha} \in \mathbb{R}^d : \quad \hat{D}(\boldsymbol{\alpha}) \le c\}$ *is compact for all* $c \ge 0$*.*

Theorem 1([2]). *Assume*

$$\liminf_{T\to\infty} \frac{1}{T} \ln \mathbf{P}(\tau \ge T) \ge -\lambda_+.$$

Then process $\{\frac{\mathbf{Z}(T)}{T}\}$ *satisfies LDP in* $\mathbb{R}^d$ *with a good rate fuction* $\hat{D}(\boldsymbol{\alpha})$*, also*

$$\lim_{T\to\infty} \frac{1}{T} \ln \mathbf{E} e^{\boldsymbol{\mu}\mathbf{Z}(T)} = -\hat{A}(\boldsymbol{\mu}).$$

If, in additionally,

$$\lambda_+ \ge D(\mathbf{0})$$

hold, then

$$\hat{A}(\boldsymbol{\mu}) = A(\boldsymbol{\mu}), \quad \hat{D}(\boldsymbol{\alpha}) = D(\boldsymbol{\alpha}),$$

and condition (3) can be omitted.

In case $d = 1$, $\lambda_+ \ge D(\mathbf{0})$, Theorem 1 was established in [1].

4. The exact asymptotics in regular case for $\mathbf{Z}(T)$ and $\mathbf{Y}(T)$

Let us define $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\alpha}) : \ \mathbb{R}^d \to \mathbb{R}^d$ so that

$$D(\boldsymbol{\alpha}) = \sup_{\boldsymbol{\mu}}\{\boldsymbol{\mu}\boldsymbol{\alpha} - A(\boldsymbol{\mu})\} = \boldsymbol{\mu}(\boldsymbol{\alpha})\boldsymbol{\alpha} - A(\boldsymbol{\mu}(\boldsymbol{\alpha})).$$

Then $(\lambda(\boldsymbol{\alpha}), \boldsymbol{\mu}(\boldsymbol{\alpha})) := (-A(\boldsymbol{\mu}(\boldsymbol{\alpha})), \boldsymbol{\mu}(\boldsymbol{\alpha}))$ belongs to the boundary $\partial\mathcal{A}^{\le 0}$ of the set $\mathcal{A}^{\le 0}$. Let put

$$\mathfrak{A} := \{\boldsymbol{\alpha} \in \mathbb{R}^d : \ (\lambda(\boldsymbol{\alpha}), \boldsymbol{\mu}(\boldsymbol{\alpha})) \in (\mathcal{A})\}, \quad \mathfrak{C} := \{\boldsymbol{\alpha} \in \mathbb{R}^d : \ \boldsymbol{\mu}(\boldsymbol{\alpha}) \in (\mathcal{M})\},$$

where

$$\mathcal{A} := \{(\lambda, \boldsymbol{\mu}) : \ A(\lambda, \boldsymbol{\mu}) < \infty\}, \quad \mathcal{M} := \{\boldsymbol{\mu} : \ \mathbf{E} e^{\boldsymbol{\mu}\boldsymbol{\zeta}} < \infty\}.$$

Put

$$\mathfrak{T} := \{\boldsymbol{\alpha} : \ \lambda(\boldsymbol{\alpha}) \ge \lambda_+\}.$$

Theorem 2.([3]) *Let*

$$\boldsymbol{\alpha} := \frac{\mathbf{x}}{T} \to \boldsymbol{\alpha}^0 \quad as \quad T \to \infty.$$

I. If $\boldsymbol{\alpha}^0 \in \mathfrak{A} \setminus \mathfrak{T}$*, then*

$$\mathbf{P}(\mathbf{Z}(T) \in \Delta[\mathbf{x}]) = \frac{\Delta^d}{T^{\frac{d}{2}}} I_{\mathbf{Z}}(\boldsymbol{\alpha}) e^{-TD(\boldsymbol{\alpha})}(1 + o(1)).$$

II. If $\boldsymbol{\alpha}^0 \in \mathfrak{A} \cap \mathfrak{C}$, *then*

$$\mathbf{P}(\mathbf{Y}(T) \in \Delta[\mathbf{x}]) = \frac{\Delta^d}{T^{\frac{d}{2}}} I_{\mathbf{Y}}(\boldsymbol{\alpha}) e^{-TD(\boldsymbol{\alpha})}(1+o(1)),$$

where the continuous functions $I_{\mathbf{Z}}(\boldsymbol{\alpha})$, $I_{\mathbf{Y}}(\boldsymbol{\alpha})$ *are known in explicit form.*

Theorem 2 in case $d = 1$ was established in [1].

5. The exact asymptotics in non-regular case for $\mathbf{Z}(T)$

$[\mathbf{F}_\tau]$. *For all* $t > 0$

$$\mathbf{P}(\tau \geq t) = L(t) e^{-\lambda_+ + ct^\gamma},$$

where $\gamma \in [0,1)$, $c \in \mathbb{R}$, $L(t) = t^\beta l(t)$*—regularly varying as* $t \to \infty$ *function.*

Theorem 3. ([3]) *Assume* $[\mathbf{F}_\tau]$ *holds. Let* $\kappa := [\frac{1}{1-\gamma}]$; $\boldsymbol{\alpha}^0 \in \mathfrak{A} \cap (\mathfrak{T})$, $\boldsymbol{\alpha}^0 \neq \mathbf{0}$. *Then for* $\boldsymbol{\alpha} := \frac{\mathbf{x}}{T} \to \boldsymbol{\alpha}^0$ *we have*

$$\mathbf{P}(\mathbf{Z}(T) \in \Delta[\mathbf{x}]) =$$

$$\frac{\Delta^d L(T)}{T^{\frac{(d-1)}{2}}} C(\boldsymbol{\alpha}) e^{-T\hat{D}(\boldsymbol{\alpha}) + \sum_{k=1}^{\kappa} T^{k\gamma-(k-1)} g_k(\boldsymbol{\alpha})}(1+o(1)),$$

where the continuous functions $C(\boldsymbol{\alpha}), g_1(\boldsymbol{\alpha}), \cdots, g_\kappa(\boldsymbol{\alpha})$ *are known in explicit form.*

6. The exact asymptotics of finite-dimensional distributions for $\mathbf{Z}(T)$

Let

$$U_0 = 0, \quad U_1 \to \infty, \cdots, U_m \to \infty; \quad W_j := U_0 + \cdots + U_j.$$

In [3] were found fairly extensive conditions, under which for

$$\boldsymbol{\alpha}_0 := \mathbf{0}, \quad \boldsymbol{\alpha}_j := \frac{\mathbf{x}_j}{U_j} \to \boldsymbol{\alpha}_j^0, \quad j = 1, \cdots, m,$$

holds

$$\mathbf{P}(\cap_{j=1}^m \{\mathbf{Z}(W_j) - \mathbf{Z}(W_{j-1}) \in \Delta_j[\mathbf{x}_j)\}) =$$

$$\prod_{j=1}^m \frac{\Delta_j^d}{U_j^{\frac{d}{2}}} I_{\mathbf{Z}}(\boldsymbol{\alpha}_{j-1}, \boldsymbol{\alpha}_j) e^{-U_j D(\boldsymbol{\alpha}_j)}(1+o(1)).$$

In article [1], in case $d = 1$ proposed conditions, under which (6) holds.

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