UDC 519.21

Asymptotic behavior of reliability function for multidimensional aggregated Weibull type reliability indices

Yu. Farkas*, E. Hashorva*, V. I. Piterbarg†

* *Faculty of busyness and economics, Lausanne university, Dorigny CH-1015 Lausanne, Switzerland.*

† *Department of Probability Theory, Moscow State University, Leninskie Gory 1, Moscow, 119991, Russia*

Abstract. We study asymptotic behavior of an multidimensional reliability function (ruin probability) for multidimensional aggregated reliability index which is a linear combination of single indexes distributed by Weibull like laws.

Keywords: Total reliability index, dependent reliability indexes; Weibull-like reliability indexes, multidimensional ruin probability, multidimensional distribution tails.

1. Introduction

We study asymptotic behavior of the multidimensional reliability function, see book [1],

$$P(u;\mathbf{c},\Lambda) := \mathbf{P}(Q_{n,m} \ge c_m u, \quad m = 1, ..., d),$$

as $u \to \infty$, for multidimensional aggregated reliability index $\{Q_{n,m},\ m = 1, ..., d\}$, components of vector $\mathbf{c} = \{c_m,\ m = 1, ..., d\}$, are positive coefficients of proportionality of the indexes. Reliability indices $Q_{n,m}$ are defined as weighted sums $Q_{n,m} = \sum_{i=1}^{n} \lambda_{i,m} X_i$ of basic indices X_i, which are modeled by independent random variables having Weibull like tail distributions

$$\mathbf{P}\left(X_i \ge x\right) = g_i(x)e^{-x^p} = x^{\alpha_i}\ell_i(x)e^{-x^p},\ x \ge 0,$$

where $p > 1$ and $\ell_i(x)$, $i = 1, ..., n$, slowly varying on infinity. The matrix $\Lambda := \{\lambda_{i,m}, i = 1, ..., n,\ m = 1, ..., d\}$ is the matrix of factors of the risks being shared. We assume non-degeneracy of the aggregated reliability index,

$$d \le n \text{ and } \operatorname{rank}\Lambda = d,$$

otherwise some indexes are linear combinations of other ones. Such schemes are also used in financial and actuarial models for study ruin probabilities of financial/actuarial portfolios.

From theory of slowly varying functions, see book [2], it follows that we may and do assume in case when the densities exist, that, with some other $\ell_i(x)$,

$$f_i(x) = \mathbf{I}_{\{x \geq 0\}} p x^{\alpha_i + p - 1} \ell_i(x) e^{-x^p}, \ i = 1, ..., n.$$

In case $P(u; \mathbf{c}, \Lambda)$ is absolutely continuous in u it makes a sense to introduce the density of multidimensional reliability

$$p(u; \mathbf{c}, \Lambda) := -\frac{d}{du} P(u; \mathbf{c}, \Lambda).$$

In case $d = 1$, $p(u; \mathbf{c}, \Lambda)$ is simply the probability density of the reliability index.

The main tools of studies are Laplace saddle point asymptotic method and Lemma of consistency of ruin probability.

2. One-dimensional case

We need a notation of shortened reliability indexes with some other X_i's, with the same as above assumptions.

$$Q'_{k,m} = \sum_{i=1}^{k} \lambda_{i,m} X_i', \quad m = 1, ..., d, \ k = 1, ..., n.$$

Assume that

$$\lim_{x \to \infty} \frac{\mathbf{P}(X_i' \geq x)}{\mathbf{P}(X_i \geq x)} = 1, \ i = 1, ..., n.$$

Lemma 1 *For any arbitrarily small* $\delta > 0$,

$$\mathbf{P}(Q_{n,m} \geq c_m u, \quad m = 1, ..., d)$$

$$= (1 + o(1)) \mathbf{P}(Q'_{n,m} \geq c_m u, \quad m = 1, ..., d) + R(u; \delta),$$

as $u \to \infty$, *with*

$$R(u; \delta) = O\left(\sum_{k=1}^{n-1} \mathbf{P}(Q'_{k,m} \geq (1 - \delta) c_m u, m = 1, ..., d) \right)$$

as $u \to \infty$.

In this section we evaluate the asymptotic behavior of

$$P(u; \Lambda) := \mathbf{P}(\lambda_1 X_1 + ... + \lambda_n X_n \geq u)$$

as $u \to \infty$.

This problem is not new, it has been solved by several authors. Very simple way can be performed, for example, using induction, a lemma of Piterbarg, see book [6], and the above consistency Lemma 1.

Taking in mind multidimensional case, in order to show our geometrical approach, we show in this simple case how to evaluate this asymptotic by Laplace asymptotic method. Besides, some steps of the proof are used in the case of d-dimensional case.

First assume that the risks X_i, $i = 1, ..., n$, have densities:

$$\mathbf{P}(Q_n \geq u) = u^n \int_{\mathcal{C}} \prod_{i=1}^{n} g_i'(uy_i) e^{-u^p ||\mathbf{y}||_p^p} d\mathbf{y},$$

where $g_i'(x) := px^{\alpha_i+p-1}\ell_i(x)$ is described above, and

$$\mathcal{C} := \left\{ \sum_{i=1}^{n} \lambda_i y_i \geq 1, y_i \geq 0, i = 1, ..., n \right\}.$$

The main steps of applying here Laplace asymptotic method are as following:

— First find the minimum of $||\mathbf{y}||_p := (\sum_{i=1}^n y_i^p)^{1/p}$ and the point of minimum on $\mathcal{C}$.
— Find the behavior of $||\mathbf{y}||$ at the point of minimum.
— Find an "informative" small compact subset of $\mathcal{C}$.
— Finally compute an elementary integral.

Now just use Lemma 1 to pass from the absolutely continuous case to the general one.

Theorem 1 *Let $X_1, ..., X_n$ be independent reliability indexes with tails (1) and λ_i, $i = 1, ..., n$, are positive weights, then*

$$P(u; \Lambda) = \frac{(1 + o(1))(2\pi)^{(n-1)/2} p^{n/2-1/2}}{(p-1)^{n/2-1/2} \sigma^{\alpha+(n-1)p/2+1/2}} \times$$
$$\times \prod_{i=1}^{n} \left(\lambda_i^{\frac{\alpha_i + p/2}{p-1}} \ell_i(u) \right) u^{\alpha+(n-1)p/2} e^{-\sigma^{1-p} u^p}$$

as $u \to \infty$, where $\alpha = \sum_{i=1}^n \alpha_i$, and $\sigma = \sum_{i=1}^n \lambda_i^{p/(p-1)}$.

Further, if the indexes have densities, then for the ruin density one has,

$$p(u; \Lambda) = -\frac{d}{du} P(u; \Lambda) = (1 + o(1)) p \sigma^{1-p} u^{p-1} P(u; \Lambda)$$

as $u \to \infty$.

3. Multidimensional case

Now we study the asymptotic behavior of multidimensional ruin probability. We have,

$$\mathbf{P}(Q'_{n,m} > c_m u, \quad m = 1, ..., d) = \int_{u\mathcal{A}} \prod_{i=1}^{n} g'_i(x_i) e^{-\sum x_i^p} d\mathbf{x}$$
$$= u^n \int_{\mathcal{A}} \prod_{i=1}^{n} F_i(uy_i) e^{-||\mathbf{y}||_p^p u^p} d\mathbf{y},$$

with

$$\mathcal{A} := \left\{ \sum_{i=1}^{n} \lambda_{i,m} y_i \geq c_m, m = 1, ..., d, \ y_i \geq 0, i = 1, ..., n \right\}.$$

We assume immediately that the risks have densities. It can be shown that even in case $d = 2$ and $p = 2$, for some c_1, c_2, the residual in Consistency Lemma 1 has the same exponential order as the ruin probability.

Now we again are in a position to study the asymptotic behavior as $u \to \infty$ of the Laplace type integral.

Several remarks on the evaluation of the asymptotic behavior of the above integral, see paper [5]

— In multidimensional case we are looking for the minimum of $||\mathbf{y}||_p$ subject to $\mathbf{y} \in \mathcal{A}$. Denote the point of minimum by $\mathbf{y}_0$, it can lie on a hyperplane of dimension $k \in [0, ..., d-1]$ which is the intersection of hyperplanes with equality signs in the description of $\mathcal{A}$. Denote by $\mathcal{I} \subset [1, ...d]$, the corresponding set of indices, so that $|\mathcal{I}| = k$. We see that this first Optimization problem is more complicated than for $d = 1$. We use here Lagrange multipliers.

— Now we use Taylor expansion at $\mathbf{y}_0$ to reduce our asymptotic study to an integral with the quadratic form under the exponent. This Optimization problem 2 is also complicated! But we may use here the results of E. Hashorva and J. Hüsler mentioned on Gaussian multidimensional tails, see paper [4].

— So, we have here two optimization problems, initially for Weibull like density and then for Gaussian density.

This way leads to the main theorem on the asymptotic behavior of the multidimensional ruin probability.

We formulate the main result for ruin probability of several portfolios of Weibull like reliability indexes.

Theorem 2 *Let $X_1, ..., X_n$, $n \geq d$, be independent Weibull like random variables (reliability indexes) having probability densities. Let the matrix Λ has rank d. Then for the ruin probability (reliability function) we have:*

1. *In case* $|\mathcal{I}| > 1$,

$$P(u;\mathbf{c},\Lambda) = \frac{(1+o(1))(2\pi)^{(n-|I|)/2}P_I}{(p(p-1))^{n/2}\sqrt{\det R_{II}}}\prod_{i=1}^{n} y_{0,i}^{1-p/2}u^{n-\frac{1}{2}np+\alpha-\frac{1}{2}|I|p}\times$$

$$\times\prod_{i=1}^{n}\ell_i(u)\exp\left(-\frac{u^p}{2}\left(\frac{p-2}{p-1}||\mathbf{y}_0||_p^p+\left\langle\hat{\mathbf{c}}_I, R_{II}^{-1}\hat{\mathbf{c}}_I\right\rangle\right)\right)$$

as $u\to\infty$. *Above we denoted* $\alpha=\sum_{i=1}^{n}\alpha_i$, $\mathbf{y}_0=(y_{0,i}, i=1,...,n)$ *is the solution to Optimization Problem 1,* $\hat{\mathbf{c}}=(\hat{\mathbf{c}}_I,\hat{\mathbf{c}}_J)$ *is the solution to Optimization Problem 2, the index sets* $I, J\subset\mathcal{I}$, *the constant* P_I, *and the matrix* R_{II} *are defined in Hashorva-Hüsler Theorem.*

2. *In case* $|\mathcal{I}|=1$, *that is,* $\mathcal{I}=\{m_0\}$, *the assertion of the above one-dimensional Theorem takes place with changing* λ_i *on* $\lambda_{i,m_0}/c_{m_0}$ *or* u *on* $c_{m_0}u$.

3. *Further, for the ruin density,*

$$p(u;\mathbf{c},\Lambda)=(1+o(1))p||\mathbf{y}_0||_p^p u^{p-1}P(u;\mathbf{c},\Lambda)$$

as $u\to\infty$.

4. Examples

1. The case $d=2$ can be considered in details, here $|\mathcal{I}|=1$ or 2. Therefore the second optimization problem is reduced to consideration of one-dimensional and two-dimensional Gaussian distributions.

2. Permutation symmetric Gaussian like ($p=2$) indexes also can be considered. It means that the asymptotic behavior of the ruin probability does not depend on their permutations. In Gaussian case it simply means that all non-diagonal elements of the covariance matrix are equal one to another.

References

1. *Gnedenko B. V., Belyaev Ju. K., Solovyov A. D.* Mathematical methods in the theory of reliability. — M.: Nauka, 1965.
2. *Bingham N. H., Goldie C. M., Teugels J. L.* Regular variation. — Cambridge University Press, 1987.
3. *Hashorva, E., Hüsler, J.* On multivariate Gaussian tails // Ann. Inst. Statist. Math. — 2003. — Vol. 55, no. 3. — P. 507–522.
4. *Hashorva, E., Hüsler, J.* — On Asymptotics of Multivariate Integrals with Applications to Records // Stochastic Models. — 2002. — Vol. 18, no. 1. — P. 41–69.

5. *Korshunov D. A., Piterbarg V. I., Hashorva E.* On the asymptotic Laplace method and its application to random chaos // Mathematical Notes. — 2015. — Vol. 97, no. 5-6. — P. 878–891.
6. *Piterbarg V. I.* Asymptotic Methods in Theory of Gaussian Random Processes and Fields. — AMS, 2012. — Vol. 148.