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Analysis of Two-Heterogeneous Server Queueing System

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Abstract. This study mainly concerned with the K-capacity queueing system with recurrent input and two heterogeneous servers. Arriving customers choose server from the empty servers with equal probability. At an arrival time the customer joins the queue if both servers are busy. In addition, an arrival leaves without service when the system capacity is achieved. The defined system is represented by semi-Markov process and embedded Markov chain is obtained. Steady-state probabilities are found and loss probability is calculated by analyzing stream of overflows.

Keywords: embedded Markov chain, heterogeneous servers, finite capacity queue, loss probability, stream of overflows.

1. Introduction

Heterogeneous server queueing models were first studied by [1]. [2], [3], [4], [5], [6], [7] are some examples of the literature on heterogeneous server queueing models. The analysis of $GI/M/n/n$ queueing loss system was presented by [8]. They analyzed the stream of overflows and obtained Laplace-Stieltjes transform of the distribution of stream of overflows are presented. In addition, they formulated the steady-state probabilities by using embedded Markov chain of the semi-Markov process. In this study, by using the results given by [8], the loss probability is calculated for the $GI/M/2/K$ queueing system. This study is an extension of [8] in that, it is assumed in this study there is a waiting space and there are two servers. Model description and assumptions are introduced in the following section. In section 3, the semi-Markov process representing the system is constructed and the analysis of the model is presented. The steady-state probabilities and loss probability are obtained in section 4. Finally discussion and conclusions are presented.

2. Model description and assumptions

Let $t_0, t_1, \ldots$ be the arrival times of the customers, where $t_0 < t_1 < \cdots$. The interarrival times are independent identically distributed with distribution function $F(t)$ and $\alpha = \int_0^\infty [1 - F(t)]dt < \infty$. The service time of each customer in server k is a random variable represented by η_k and has an exponential distribution with parameter μ_k $(k = 1, 2)$. An arriving customer may choose any one of the free servers with equal probability. When all servers are busy, customers join the queue. The system capacity is K. When the system capacity is reached, an arriving customer leaves the system without taking any service.

Let $X(t)$ be the number of customers at time t and $X_n = X(t_n - 0), n \geqslant 1$. X_n is the number of customers being in the system at the time of the n-th arrival. The semi-Markov process that represents the system is defined as follows:

$$\xi(t) = X_n, \ t_n \leqslant t < t_{n+1}, \quad n \geqslant 1.$$

The kernel of the process $\{\xi(t), t \geq 0\}$ is

$$Q_{ij}(x) = P\{(X_{n+1} = j, t_{n+1} - t_n < x) | X_n = i\}, \tag{1}$$

for all $x \geqslant 0$ and $0 \leq i, j \leq K$. For each state i, j the kernel function given by (1) can be written as follows:

$$Q_{00}(x) = \frac{1}{2}\int_0^x [(1 - e^{-\mu_1 t}) + (1 - e^{-\mu_2 t})]dF(t),$$

$$Q_{01}(x) = \frac{1}{2}\int_0^x (e^{-\mu_1 t} + e^{-\mu_2 t})dF(t),$$

$$Q_{10}(x) = \int_0^x (1 - e^{-\mu_1 t})(1 - e^{-\mu_2 t})dF(t),$$

$$Q_{11}(x) = \int_0^x [e^{-\mu_1 t}(1 - e^{-\mu_2 t}) + e^{-\mu_2 t}(1 - e^{-\mu_1 t})]dF(t),$$

$$Q_{12}(x) = \int_0^x e^{-(\mu_1 + \mu_2)t} dF(t),$$

for $i + 1 \geq j \geq 2$ and $i \geq 2$

$$Q_{ij}(x) = \int_0^x \frac{[(\mu_1 + \mu_2)t]^{i+1-j}}{(i + 1 - j)!} e^{-(\mu_1+\mu_2)t} dF(t),$$

for $i + 1 > 2$ and $j = 0$

$$Q_{ij}(x) = \int_0^x [\int_0^t \frac{(\mu_1 + \mu_2)^{i+1-2}}{(i - 2)!} y^{i-2} e^{-(\mu_1+\mu_2)y} \\ (1 - e^{-\mu_1(t-y)})(1 - e^{-\mu_2(t-y)})dy]dF(t),$$

for $i + 1 > 2$ and $j = 1$

$$Q_{ij}(x) = \int_0^x [\int_0^t \frac{(\mu_1 + \mu_2)^{i+1-2}}{(i - 2)!} y^{i-2} e^{-(\mu_1+\mu_2)y} [e^{-\mu_1(t-y)} \\ (1 - e^{-\mu_2(t-y)}) + e^{-\mu_2(t-y)}(1 - e^{-\mu_1(t-y)})]dy]dF(t),$$

$$Q_{ij}(x) = 0, j \geq i+1,$$

Let $q_{ij}(s)$ represent the Laplace-Stieltjes transform of $Q_{ij}(x)$ such that:

$$q_{ij}(s) = \int_0^{\infty} e^{-sx} dQ_{ij}(x), 0 \leq i, j \leq K(Re\{s\} \geq 0).$$

Hence $q(s) = [q_{ij}(s)]_0^K$ is obtained in matrix form as follows:

$$q(s) = \begin{pmatrix} q_{00}(s) & q_{01}(s) & 0 & 0 & \cdots & 0 \\ q_{10}(s) & q_{11}(s) & q_{12}(s) & 0 & \cdots & 0 \\ q_{20}(s) & q_{21}(s) & q_{22}(s) & q_{23}(s) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{K-1,0}(s) & q_{K-1,1}(s) & q_{K-1,2}(s) & q_{K-1,3}(s) & \cdots & q_{K-1,K}(s) \\ q_{K-1,0}(s) & q_{K-1,1}(s) & q_{K-1,2}(s) & q_{K-1,3}(s) & \cdots & q_{K-1,K}(s) \end{pmatrix} \tag{2}$$

From (2) $q(s)$ is a lower Hessenberg matrix. Let the transition probabilities are defined as $p_{ij} = P\{X_{n+1} = j | X_n = i\}$ and $P = [p_{ij}]_0^K$. Hence p_{ij} for each i, j can be obtained from the equation $p_{ij} = q_{ij}(0)$.

3. Steady-state probabilities and the loss probability

The stream of overflow analysis for $GI/M/n/n$ heterogeneous-server queueing system without waiting space is presented by [8] and both the steady-state probabilities and the loss probability are obtained as a function of the transition probabilities. Based on the results given by [8], the steady-state probabilities and the loss probability for $GI/M/2/K$ queueing system are obtained respectively as follows:

$$P_n = \frac{D_{nn}(0)}{D(1,1,...,1)}, \qquad n = 1, 2, ..., K.$$

$$P_{loss} = \frac{p_{01}p_{12}...p_{K-1,K}}{D(1,1,...,1)}, \tag{3}$$

where $D_{nn}(0)$ are the cofactors of the (n, n)th entries of matrix $[I - q(0)]$ and

$$D(1,1,...,1)=\begin{vmatrix} 1 & -p_{01} & 0 & \cdots & 0 \\ 1 & 1-p_{11} & -p_{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -p_{K-1,1} & -p_{K-1,2} & \cdots & -p_{K-1,K} \\ 1 & -p_{K-1,1} & -p_{K-1,2} & \cdots & 1-p_{K-1,K} \end{vmatrix}$$

By using (3) the loss probability for the $GI/M/2/4$ queueing model is obtained as following:

$$P_{loss}=\frac{p_{01}p_{12}p_{23}p_{34}}{(1-p_{11}+p_{01})A+p_{12}B+p_{01}p_{12}C},$$

where, $A=(1-p_{22})(1-p_{33}-p_{34})-p_{23}p_{32}$, $B=-p_{21}(1-p_{33}-p_{34})-p_{23}p_{31}$, $C=(1-p_{33}-p_{34}+p_{23}$, and

$$p_{01}=\frac{1}{2}[f(\mu_1)+f(\mu_2)],$$

$$p_{11}=f(\mu_1)+f(\mu_2)-2f(\mu_1+\mu_2),$$

$$p_{12}=f(\mu_1+\mu_2),$$

$$\begin{aligned} p_{21}=&\frac{\mu_1+\mu_2}{\mu_2}f(\mu_1)+\frac{\mu_1+\mu_2}{\mu_1}f(\mu_2)\\ &-\frac{(\mu_1+\mu_2)^2}{\mu_1\mu_2}f(\mu_1+\mu_2)-2(\mu_1+\mu_2)f^2(\mu_1+\mu_2),\end{aligned}$$

$$p_{22}=(\mu_1+\mu_2)f^2(\mu_1+\mu_2),$$

$$p_{23}=f(\mu_1+\mu_2),$$

$$\begin{aligned} p_{31}=&\frac{(\mu_1+\mu_2)^2}{\mu_2^2}f(\mu_1)+\frac{(\mu_1+\mu_2)^2}{\mu_1^2}f(\mu_2)\\ &-\left[\frac{(\mu_1+\mu_2)^2}{\mu_1^2}+\frac{(\mu_1+\mu_2)^2}{\mu_2^2}\right]f(\mu_1+\mu_2)\\ &-\left[\frac{(\mu_1+\mu_2)^2}{\mu_1}+\frac{(\mu_1+\mu_2)^2}{\mu_2}\right]f^2(\mu_1+\mu_2)\\ &-2(\mu_1+\mu_2)^2f^3(\mu_1+\mu_2),\end{aligned}$$

$$p_{32}=(\mu_1+\mu_2)^2f^3(\mu_1+\mu_2),$$

$$p_{33} = (\mu_1 + \mu_2) f^2(\mu_1 + \mu_2),$$

$$p_{34} = f(\mu_1 + \mu_2).$$

4. Conclusions

Heterogeneous server $GI/M/2/K$ queuing system is analyzed by semi-Markov process and embedded Markov chain of the process is obtained. The steady-state probabilities are obtained and the loss probabilities are calculated. Since the steady state probabilities are expressed using the determinant, these probabilities can be computed easily once the transition matrix is known. The calculation of the average number waiting in line and also obtaining the distribution function of the waiting time may be the further research directions.

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