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Discrete gamma probability distribution approximation in retrial queues

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Abstract. In the paper, the retrial queueing system of $MMPP/M/1$ type is considered. The process of the number of calls in the system is analyzed. We propose the method of the discrete gamma approximation. The numerical analysis of comparison of distributions obtained by simulation and approximate ones for different values of the system parameters is presented.

Keywords: retrial queueing system, MMPP, discrete gamma distribution.

1. Introduction

Retrial queueing systems (or queueing systems with repeated calls) are the new models of queueing theory widely used for study of real telecommunication systems, cellular networks, call centres [1]. Retrial queues are characterized by the feature that an unserviced call do not joint a queue and not leave the system immediately, but goes to some virtual place (orbit), then it tries to get service again after random time.

The comprehensive description and the detailed comparison of classical queueing systems and retrial queues are made by Falin and Artalejo in books [1, 2]. Asymptotic and approximate methods are also offered by Falin, Anisimov, Yang, Diamond, Aissani, etc.

In previous papers (e.g. [3]), we shown that the probability distribution of the number of calls in the orbit in various retrial queues has the gamma distribution form under heavy load condition. In addition, we proposed the gamma approximation method [4] which can be applied for more wide area (not only heavy load). Thus, in this paper, we try to improve the results of approximations by using the discrete analogue of gamma distribution.

2. Retrial queue $MMPP/M/1$

For the approximation method demonstrating, let us consider a single server retrial queueing system $MMPP/M/1$. The system structure is presented in Figure 1.

Primary calls arrive from outside at the system according to Markovian Modulated Poisson Process (MMPP) defined by matrix $\mathbf{D_0}$ and $\mathbf{D_1}$ [5, 6]. If a primary call finds the server free, it stays here with service time distributed exponentially with rate μ. Otherwise, the call goes to an orbit,

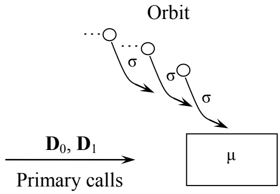

Figure 1. Retrial queueing system $MMPP/M/1$

where it stays during random time distributed by the exponential law with rate σ. After the delay, the call makes an attempt to reach the server again. If the server is free, the call gets the service, otherwise, the call instantly returns to the orbit.

The MMPP underlying process $n(t)$ is a Markov chain with continuous time and finite set of states $n = 1, 2, \ldots, W$. We introduce the generator of the process $n(t)$ as matrix $\mathbf{Q} = \mathbf{D_0} + \mathbf{D_1}$ with elements q_{mv}, where $m, v = 1, 2, \ldots, W$. The matrix $\mathbf{D_1}$ is diagonal with elements of conditional arrival rates λ_n $(n = 1, 2, \ldots, W)$. Thus, we write $\mathbf{\Lambda} = \mathtt{diag}\{\lambda_n\}$. The fundamental rate of MMPP is defined as follows $\lambda = \mathbf{r} \cdot \mathbf{\Lambda} \cdot \mathbf{e}$.

Let $i(t)$ be the number of calls in the system and $k(t)$ be the server state:

$$k(t) = \begin{cases} 0, & \text{if the server is free,} \\ 1, & \text{if the server is busy.} \end{cases}$$

Denote $P(k, n, i, t) = P\{k(t) = k, n(t) = n, i(t) = i\}$. The process $\{k(t), n(t), i(t) : t \geq 0\}$ is the multi-dimensional continuous time Markov chain. The following system of Kolmogorov equations for the stationary distribution $P(k, n, i) = \lim_{t \to \infty} P(k, n, i, t)$ is derived for $i > 0,\ n = \overline{1, W}$

$$\begin{cases} -(\lambda_n + i\sigma - q_{nn})P(0, n, i) + \mu P(1, n, i+1) + \sum_{v \neq n} P(0, v, i) q_{vn} = 0, \\ -(\lambda_n + \mu - q_{nn})P(1, n, i) + \lambda_n P(1, n, i-1) + \lambda_n P(0, n, i-1) \\ \qquad + i\sigma P(0, n, i) + \sum_{v \neq n} P(1, v, i) q_{vn} = 0. \end{cases} \tag{1}$$

Let us introduce row vectors

$$\mathbf{P}_k(i) = \{P(k, 1, i), P(k, 2, i), \ldots, P(k, W, i)\}.$$

By $\mathbf{H}_k(u) = \sum_i e^{jui}\mathbf{P}_k(i)$ denote the partial characteristic functions, where $k = 0, 1$ and $j = \sqrt{-1}$. Then equations (1) have the following matrix form:

$$\begin{cases} \mathbf{H}_0(u)\left(\mathbf{Q} - \mathbf{\Lambda}\right) + j\sigma\mathbf{H}_0'(u) + \mu e^{-ju}\mathbf{H}_1(u) = \mathbf{0}, \\ \mathbf{H}_1(u)\left(\mathbf{Q} - \mathbf{\Lambda}(1 - e^{ju}) - \mu\mathbf{I}\right) + e^{ju}\mathbf{H}_0(u)\mathbf{\Lambda} - j\sigma\mathbf{H}_0'(u) = \mathbf{0} \end{cases} \tag{2}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{e} = \{1, 1, \ldots, 1\}^T$ and $\mathbf{0} = \{0, 0, \ldots, 0\}$.

The characteristic function of the number of calls in the system is defined as follows

$$h(u) = (\mathbf{H}_1(u) + \mathbf{H}_0(u))\mathbf{e}.$$

From system (2), we can not obtained the exact formula for $h(u)$. Thus, we propose the approximation method.

3. Discrete gamma approximation

In the previous papers, we proposed the Gaussian, quasi-geometric and gamma approximation methods for retrial queues [4,7]. Here, we offer new type approximation by the discrete analogue of gamma distribution.

Definition. By the discrete gamma distribution we call a discrete probability distribution $Pg(i)$ for $i \geq 0$, which characteristic function has the following form

$$G(u) = \left(\frac{1 - \gamma}{1 - \gamma e^{ju}}\right)^{\alpha}, \tag{3}$$

with parameters $\alpha > 0$ and $0 < \gamma < 1$.

It is easy to show that the parameters α and γ are expressed in terms of the mean E and the variance var of the distribution $Pg(i)$ as follows

$$\gamma = 1 - \frac{E}{var}, \quad \alpha = E \cdot \frac{1 - \gamma}{\gamma}.$$

The method of the approximation consists in approximating by the discrete gamma distribution $Pg(i)$ which parameters are calculated via the known mean and the variance.

Note that mean and variance of the distribution $P(i)$ can be calculated approximately using some analytical methods or numerical algorithms.

We offer this type of approximation because the characteristic function of number of calls in the retrial queue $M/M/1$ (a particular case of considered model) has the form (3).

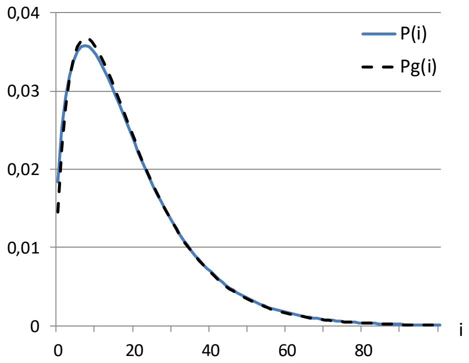

Figure 2. Comparison of the approximate (dashed line) and the empiric (solid line) distributions for $\sigma = 1$ and $\rho = 0.9$

4. Numerical analysis

Let us we present some numerical examples to demonstrate the applicability area of the approximation. We perform the system evolution simulation using software platform ODIS [9], which realizes a discrete-event simulation approach, and we compare statistical results with analytical ones derived in the paper.

In the example, let the service rate be $\mu = 1$, the arrival process be MMPP with 3 states and following parameters

$$\mathbf{\Lambda} = \begin{bmatrix} 0.364 & 0 & 0 \\ 0 & 0.727 & 0 \\ 0 & 0 & 1.091 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} -0.5 & 0.4 & 0.1 \\ 0.2 & -0.5 & 0.3 \\ 0.1 & 0.2 & -0.3 \end{bmatrix}.$$

It is holds that $\mathbf{r\Lambda e} = \mu = 1$ for these parameters. Thus, the system load $\rho = \mathbf{r\Lambda e}/\mu$ has values $0 < \rho < 1$.

Let us compare the probability distribution of the number of calls in the retrial queueing system $P(i)$ calculated via simulation and its approximation $Pg(i)$ with moments obtained by the method of initial moment [8].

The comparison of the distributions is shown in Figures 2. For the analysis, we use Kolmogorov distance between respective distribution functions (in Table 1).

From Table 1, we see that the Kolmogorov distance between distributions $d \leq 0.035$ for the wide range of σ and ρ values. Note that we obtained similar results for other values of the MMPP parameters.

Table 1

Kolmogorov distances d for various values of the parameter ρ and σ

	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 10$
$\rho = 0.3$	0.0053	0.0067	0.0185	0.0211
$\rho = 0.5$	0.0048	0.0055	0.0228	0.0300
$\rho = 0.7$	0.0030	0.0034	0.0202	0.0350
$\rho = 0.9$	0.0020	0.0030	0.0091	0.0224

5. Conclusions

In this regard, the retrial queueing system of $MMPP/M/1$ type is considered in the paper. The method of the discrete gamma approximation for probability distribution of the number of calls in the system is offered. The numerical comparison of exact and approximate distributions for different values of the system parameters shows the wide range of the method application.

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