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Asymptotic Analysis of Queueing Models based on Synchronization Method

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Abstract. This paper is focused on the stability conditions of the multiserver queueing system with heterogeneous servers and a regenerative input flow $X(t)$. The main idea is constructing an auxiliary service process $Y(t)$ which is also a regenerative flow and defining the common points of regeneration for the both processes $X(t)$ and $Y(t)$. Then the traffic rate of the system is defined in terms of the mean of the increments of these processes on the common regeneration period. It allows to use well-known results from the renewal theory to find the instability and stability conditions. The possibilities of the proposed approach are demonstrated by examples.

Keywords: regenerative flow, synchronization, stability condition, service discipline.

1. Model description

We consider a multiserver queueing system with heterogeneous servers and a regenerative input flow. We assume that the input flow is a regenerative one for three reasons. Firstly, a process describing the performance of the system under some natural assumptions turns out to be a classical regenerative process [4, 11] and the renewal theory gives very effective tools for asymptotic analysis of the system. Secondly, the class of regenerative flows is rather wide. It includes recurrent, semi-Markov, Markov-modulated, Markov-arrival flows and others [2]. Finally, a regenerative flow has some useful properties that make it possible to investigate various applied models.

Our aim is asymptotic analysis of the multiserver queueing system with a regenerative input flow $X(t)$ with points of regeneration $\{\theta_n^{(1)}\}_{n=1}^{\infty}(\theta_0^{(1)} = 0)$ basing on synchronization of this flow and an auxiliary service flow $Y(t)$. This process is the number of customers which can be served up to time t under assumption that there are always customers for service. We consider the discrete-time as well as continuous-time queueing systems (see [10]) and assume that $Y(t)$ is a regenerative flow with points of regeneration $\{\theta_n^{(2)}\}_{n=1}^{\infty}(\theta_0^{(2)} = 0)$ and this flow does not depend on $X(t)$. For the discrete-time case $Y(t)$ is an aperiodic regenerative flow, i.e. regeneration period $\tau_n^{(2)} = \theta_n^{(2)} - \theta_{n-1}^{(2)}$ has an aperiodic distribution

$$GCD\{k : P(\tau_n^{(2)} = k)\} = 1 \tag{1}$$

(GCD - the greatest common divisor) and for the continuous-time case $Y(t)$ is a strongly regenerative flow. This means that the regeneration period $\tau_n^{(2)}$ has a form

$$\tau_n^{(2)} = v_n^{(1)} + v_n^{(2)} \tag{2}$$

where $P(v_n^{(1)} > x) = e^{-\delta x}(\delta \in (0,\infty))$, $v_n^{(1)}$ and $v_n^{(2)}$ are independent random variables and $Y(\theta_{n-1}^{(2)} + v_n^{(1)}) = Y(\theta_{n-1}^{(2)})$.

We also assume that $\mathbf{E}\theta_1^{(1)} < \infty, \mathbf{E}X(\theta_1^{(1)}) < \infty, \mathbf{E}\theta_1^{(2)} < \infty, \mathbf{E}Y(\theta_1^{(2)}) < \infty$. Then the rates λ_X and λ_Y of these flows are limits w.p.1 $\lambda_X = \lim\limits_{t\to\infty} \frac{X(t)}{t} = \frac{\mathbf{E}X(\theta_1^{(1)})}{\mathbf{E}\theta_1^{(1)}}$, $\lambda_Y = \lim\limits_{t\to\infty} \frac{Y(t)}{t} = \frac{\mathbf{E}Y(\theta_1^{(2)})}{\mathbf{E}\theta_1^{(2)}}$.

2. Synchronization of $X(t)$ and $Y(t)$

We determine the common points of regeneration $\{T_n\}_{n=1}^{\infty}$ for $X(t)$ and $Y(t)$ putting in the discrete-time case
$T_n = \min\left\{\theta_j^{(1)} > T_{n-1} : \bigcup_{l=1}^{\infty}\{\theta_j^{(1)} = \theta_l^{(2)}\}\right\}, T_0 = 0$ and in the continuous-time case $T_n = \min\left\{\theta_j^{(1)} > T_{n-1} : \bigcup_{l=1}^{\infty}\{\theta_{l-1}^{(2)} < \theta_j^{(1)} \leq \theta_{l-1}^{(2)} + v_l^{(1)}\}\right\}$.

Lemma 1 *Let for the continuous-time (discrete-time) Assumption* (2) *(Assumption* (1)*) be fulfilled. Then the sequence* $\{T_n\}_{n=1}^{\infty}$ *consists of common regeneration points for* $X(t)$ *and* $Y(t)$ *and*
$\boldsymbol{E}(T_n - T_{n-1}) = \boldsymbol{E}T_1 = \delta \boldsymbol{E}\theta_1^{(1)} \boldsymbol{E}\theta_1^{(2)} < \infty$ *for the continuous-time case,*
$\boldsymbol{E}T_1 = \boldsymbol{E}\theta_1^{(1)} \boldsymbol{E}\theta_1^{(2)} < \infty$ *for the discrete-time case.*
The proof is based on the renewal theory and Blackwell's theorem [4, 11].

Let $\Delta_Y(n) = Y(T_n) - Y(T_{n-1})$, $\Delta_X(n) = X(T_n) - X(T_{n-1})$. Then $\lambda_X = \frac{E\Delta_X(n)}{E(T_n - T_{n-1})}$, $\lambda_Y = \frac{E\Delta_Y(n)}{E(T_n - T_{n-1})}$. therefore the traffic rate

$$\rho = \frac{\lambda_X}{\lambda_Y} = \frac{E\Delta_X(n)}{E\Delta_Y(n)}.$$

We define the stochastic flow $\tilde{Y}(t)$ as the number of customers really served at the system during time interval $[0,t)$.

Condition 1 *The following stochastic inequalities take place*
$\tilde{\Delta}_Y(n) = \tilde{Y}(T_n) - \tilde{Y}(T_{n-1}) \leq \Delta_Y(n)$, $(n = 1, 2, ...)$.

Let $Q(t)$ be the number of customers at the system including the customers on the service at time, i.e.

$$Q(t) = Q(0) + X(t) - \tilde{Y}(t).$$

Condition 2 *There are two possible cases:*
(i) $Q(t)$ *is a stochastically bounded process, i.e.* $\lim\limits_{x\to\infty}\liminf\limits_{t\to\infty} P(Q(t) \leq x) = 1$;
(ii) $Q(t) \xrightarrow[t\to\infty]{P} \infty$.

Let us define the event $A_n = \{Q(t) \geq m \text{ for all } t \in [T_n, T_{n+1}]\}$.

Condition 3 *If Condition 2 (ii) takes place then for any* $\epsilon > 0$ *there is* n_ϵ *such that* $E\tilde{\Delta}_Y(n)\mathbb{I}(A_n) \geq E\Delta_Y(n) - \epsilon$ *for* $n > n_\epsilon$.

Here $\mathbb{I}(A_n)$ *is an indicator function for* A_n.

3. Stability and instability results

Theorem 1 *Let Condition 1 be fulfilled. If* $\rho \geq 1$ *then* $Q(t) \xrightarrow[t\to\infty]{P} \infty$.

To proof Theorem 1 we introduce the embedded process $Q_n = Q(T_n - 0)$ and use the well known results for random walks [9].

Theorem 2 *Let Conditions 2 and 3 be fulfilled. If* $\rho < 1$ *then* $Q(t)$ *is a stochastically bounded process.*

Let us note that under some additional conditions $Q(t)$ is a regenerative process with points of regeneration $\{T_{n_k}\}_{n=1}^{\infty}$ such that $Q(T_{n_k} - 0) = 0$ (see [3]). Then $Q(t)$ is a stable process when $\rho < 1$.

4. Heavy-traffic situation

Here we focus on the limit theorem for the process $Q(t)$ in the case $\rho \geq 1$. Denote $\xi_n^{(1)} = X(\theta_n^{(1)}) - X(\theta_{n-1}^{(1)}), \xi_n^{(2)} = Y(\theta_n^{(2)}) - Y(\theta_{n-1}^{(2)})$, $\tau_n^{(1)} = \theta_n^{(1)} - \theta_{n-1}^{(1)}$, $\tau_n^{(2)} = \theta_n^{(2)} - \theta_{n-1}^{(2)}$, $a_i = \mathbb{E}\xi_n^{(i)}, \mu_i = \mathbb{E}\tau_n^{(i)}$, $\sigma^2_{\xi^{(i)}} = Var\,\xi_n^{(i)}, \sigma^2_{\tau^{(i)}} = Var\,\tau_n^{(i)}, i = 1, 2$; and put

$$\sigma_X^2 = \frac{\sigma^2_{\xi^{(1)}}}{\mu_1} + \frac{a_1^2\sigma^2_{\tau^{(2)}}}{\mu_1^3} - \frac{2a_1 cov(\xi_n^{(1)}, \tau_n^{(1)})}{\mu_1^2}.$$

In evident notation σ_Y^2 is defined by the same formula.

Theorem 3 *Let Condition 1 be fulfilled and*

$$\mathbb{E}(\tau_1^{(i)})^{2+\delta} < \infty, \quad \mathbb{E}(\xi_1^{(i)})^{2+\delta} < \infty, \quad i = 1, 2$$

for some $\delta > 0$. *Then for the case* $\rho > 1$ *on any limit interval* $[0, h]$ *the scaled process*

$$\tilde{Q}_T(t) = \frac{Q(tT) - \lambda_Y(\rho - 1)tT}{\sqrt{T(\sigma_X^2 + \sigma_Y^2)}}$$

converges weakly to the process of Brownian motion.

If $\rho = 1$ *then*

$$\tilde{Q}_T(t) = \frac{Q(tT)}{\sqrt{T(\sigma_X^2 + \sigma_Y^2)}}$$

converges weakly to the absolute value of Brownian motion process.

The proofs of these statements are based on well known results for the renewal processes [5, 9], properties of regenerative flows [1] and ideas from [7, 8].

5. Example. Queueing system with interruptions of the service

As an example, we consider a multichannel heterogeneous queueing system with interruptions of the service (see [10]). Let $\{s_{i,n}^{(2)}\}_{n=1}^{\infty}$ be the moments of breakdowns and $\{s_{i,n}^{(1)}\}_{n=1}^{\infty}$ be the moments of restorations for the ith server. Here

$$0 = s_{i,0}^{(2)} < s_{i,1}^{(1)} < s_{i,1}^{(2)} < \ldots.$$

Then $u_{i,n}^{(1)} = s_{i,n}^{(1)} - s_{i,n}^{(2)}$ and $u_{i,n}^{(2)} = s_{i,n}^{(2)} - s_{i,n}^{(1)}$ denote the length of the nth blocked and the nth available period of the ith server respectively $(i = \overline{1,m})$. The sequence $\{u_{i,n}^{(1)}, u_{i,n}^{(2)}\}_{n=1}^{\infty}$ consists of iid random vectors (for all $(i = \overline{1,m})$) that do not depend on the input flow $X(t)$ and service times. Let $u_{i,n} = u_{i,n}^{(1)} + u_{i,n}^{(2)}$ be the length of the nth cycle for the server i. A cycle consists of a blocked period followed by an available period. We assume that $Eu_{i,n}^{(1)} = a_i^{(1)} < \infty$, $Eu_{i,n}^{(2)} = a_i^{(2)} < \infty$, $a_i = a_i^{(1)} + a_i^{(2)}$ $(i = \overline{1,m})$ and put $n_i(t) = 0$ if the ith server is in an unavailable state at time t and $n_i(t) = 1$ otherwise $(i = \overline{1,m})$.

We consider the preemptive repeat different service discipline that means that the service is repeated after restoration of the server the new service is independent of the original service time (see [6]). Service times of the customers served by the ith server $\{\eta_{in}\}_{n=1}^{\infty}$ are iid random variables and $b_i = \mathbb{E}\eta_{in} < \infty$.

Let $N_0(t) = \max\{k : \theta_k^{(1)} \le t\}$, $N_i(t) = \max\{k : s_{i,k}^{(2)} \le t\}$, $i = \overline{1,m}$, and assume that these counting processes are aperiodic ones. Then an auxiliary process $Y(t)$ introduced in Section 1 is an aperiodic regenerative flow with points of regeneration $\{\theta_n^{(2)}\}$ defining by the relations

$$\theta_n^{(2)} = \min\left\{ t > \theta_{n-1}^{(2)} : \bigcap_{i=1}^{m} [N_i(t) - N_i(t-1) > 0] \right\}, \theta_0^{(2)} = 0.$$

One may easily verify that

$$\lambda_Y = \lim_{t\to\infty} \frac{Y(t)}{t} = \sum_{i=1}^{m} \frac{\mathbb{E}H_i(u_{i,1}^{(2)})}{a_i}$$

where $H_i(t)$ is the renewal function corresponding the sequence $\{\eta_{in}\}_{n=1}^{\infty}$.

We get from Theorems 1 and 2

Corollary *For the system under consideration* *(i)* $Q(t) \xrightarrow[t\to\infty]{P} \infty$ *if* $\rho \geq 1$*;*

(ii) $Q(t)$ *is a stochastically bounded process if* $\rho < 1$*.*

Remark *This approach was also used for the stability analysis of the queueing systems with a preemptive priority discipline.*

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