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Non-classical boundary crossing problems for general random walks

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Abstract. In this paper we consider non-classical random walks and investigate asymptotic behavior of first-passage times over moving boundaries. The survey of such results and of methods for their proofs is presented. Several new results are also found.

Keywords: Random walk, first-passage time, moving boundary, Brownian motion.

1. Introduction

Consider a random walk $S_1, S_2, \ldots$ with $S_0 = 0$. For arbitrary real numbers y and $g_1, g_2, \ldots$ let

$$T_{y,g} := \min\{n \geq 1 : y + S_n \leq g_n\} \tag{1}$$

be the first crossing of the moving boundary g_n by the random walk S_n. The main purpose of the present paper is to study the asymptotic behaviour of the distributions of first-passage times over moving boundaries

$$\mathbf{P}(T_{y,g} > n) = \mathbf{P}(y + \min_{1\leq k\leq n}(S_k - g_k) > 0) \quad \text{as } n \to \infty, \tag{2}$$

for general random walks. An important particular case of this problem is the case of a constant boundary $g_n \equiv 0$.

If all $X_k = S_k - S_{k-1}, k = 1, 2, \ldots$, are independent and have identical distribution then the elegant result of Doney [6] is available for asymptotically stable random walks. In particular, if $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$ then for every fixed $y \geq 0$

$$\mathbf{P}(T_{y,0} > n) \sim \sqrt{\frac{2}{\pi}} \frac{\mathbf{E}[-S_{T_{y,0}}]}{\sqrt{n}}. \tag{3}$$

The use of the Wiener-Hopf factorization is a traditional approach to the derivation of (3). But if the increments X_k have different distributions or if we consider moving boundaries then there is no hope to generalize the factorization approach via the Wiener-Hopf identities to such random walks.

For such random walks a different approach was suggested in [2]. It is based on an idea due to Denisov and Wahtel (see [3], [4], [5] for another applications of this idea).

To describe the main result from [2] introduce assumptions:

$$\mathbf{E}X_k = 0 \quad \text{and} \quad 0 < B_k^2 := \mathbf{E}S_k^2 < \infty \quad \text{for all } k \geq 1. \tag{4}$$

We need also the *Lindeberg condition*

$$L_n^2(\varepsilon) := \frac{1}{B_n^2} \sum_{k=1}^{n} \mathbf{E}[X_k^2; |X_k| > \varepsilon B_n] \to 0 \quad \text{for every } \varepsilon > 0.$$

About real numbers $\{g_n\}$ used in definition (1) we assume that

$$g_n/B_n \to 0 \quad \text{and} \quad y + \sum_{k=1}^{n} \text{essup} X_k > g_n \quad \text{for all } n \geq 1, \tag{5}$$

where $\text{essup} X_k := \sup\{x : \mathbf{P}(X_k \geq x) > 0\}$. It is worth mentioning that assumption (5) is sufficient for the fact that $\mathbf{P}(T_{y,g} > n) > 0$ for all $n \geq 1$.

Theorem 1 *Assume that random variables* $\{X_k\}$ *are independent and that conditions* (4) – (5) *hold. Then*

$$\mathbf{P}(T_{y,g} > n) \sim \sqrt{\frac{2}{\pi}} \frac{U_{y,g}(B_n^2)}{B_n}, \tag{6}$$

where $U_{y,g}(\cdot)$ *is a positive, slowly varying function with the values*

$$0 < U_{y,g}(B_n^2) = \mathbf{E}[y + S_n - g_n; T_{y,g} > n] \sim \mathbf{E}[-S_{T_{y,g}}; T_{y,g} \leq n].$$

Asymptotic formula (6) generalizes (3) to all random walks satisfying the Lindeberg condition and to all boundaries satisfying (5). It was shown in [2] that the main results of the works [1], [7], [8] and [12] are the very particular cases of Theorem 1.

2. New results

Consider a case when an arbitrary random walk $\{S_k\}$ may be approximated by a walk $\{W(t_k)\}$ for some Wiener process $W(\cdot)$, where

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \dots \qquad \text{for all} \quad k > 0.$$

So we want to try to use a strong approximation. To formulate our next results we introduce the classical random broken line

$$s(t) = S_k + X_{k+1} \frac{t - t_k}{t_{k+1} - t_k} \quad \text{for} \quad t \in [t_k, t_{k+1}], \quad k \geq 0. \tag{7}$$

We always consider $s_n(t) := s(tt_n)/\sqrt{t_n}$ as random process defined for $t \in [0,1]$ with values in the space $C[0,1]$ of continuous functions.

Theorem 2 *Suppose that for each $n \geq 1$ we can define the random walk $\{S_k,\ k \geq 1\}$ and a Brownian motion $W_n(t),\ t \in [0,1]$, on a common probability space so that*

$$\mathbf{P}\left(\delta_n := \max_{0\le t\le 1}\left|s_n(t) - W_n(t)\right| > \varepsilon_n y_n/\sqrt{t_n}\right) = o\left(y_n/\sqrt{t_n}\right) \to 0 \qquad (8)$$

for some $\varepsilon_n \to 0$. Assume in addition that

$$y_n \to \infty, \qquad y_n/\sqrt{t_n} \to 0, \qquad G_n/y_n \to 0, \qquad (9)$$

where $G_n := \max_{1\le k\le n}|g_k|$. Then

$$\mathbf{P}(T_{y_n,g} > n) = \mathbf{P}(y_n + \min_{1\le k\le n}(S_k - g_k) > 0) \sim \sqrt{2/\pi}\, y_n/\sqrt{t_n}. \qquad (10)$$

To prove the theorem note that

$$\mathbf{P}(T_{y_n,g} > n) \le \mathbf{P}(y_n + \min_{1\le k\le n} S_k > -G_n) = \mathbf{P}\left(\min_{1\le t\le 1} s_n(t) > \frac{-y_n - G_n}{\sqrt{t_n}}\right)$$
$$\le \mathbf{P}\left(\min_{1\le t\le 1} W_n(t) > \frac{-y_n - G_n - \varepsilon_n y_n}{\sqrt{t_n}}\right) + \mathbf{P}\left(\delta_n > \varepsilon_n y_n/\sqrt{t_n}\right).$$

Similarly,

$$\mathbf{P}(T_{y_n,g} > n) \ge \mathbf{P}\left(\min_{1\le t\le 1} W_n(t) > \frac{-y_n + G_n + \varepsilon_n y_n}{\sqrt{t_n}}\right) - \mathbf{P}\left(\delta_n > \varepsilon_n y_n/\sqrt{t_n}\right).$$

On the other hand, it is known that for any $x_n > 0$

$$\mathbf{P}\left(\min_{1\le t\le 1} W_n(t) > x_n\right) = \int_{-x_n}^{x_n} (2\pi)^{-1/2} e^{x_2/2} dx \sim \sqrt{2/\pi}\, x_n$$

as $x_n \to 0$. These arguments yield (10) with

$$0 < x_n = (y_n \pm G_n \pm \varepsilon_n y_n)/\sqrt{t_n} \sim y_n/\sqrt{t_n}.$$

As a simple corollary we obtain

Theorem 3 *Suppose that random variables $\{X_k\}$ are independent and that conditions (4), (5) and (9) take place when $y = y_n$ and $\sqrt{t_n} = B_n$. Assume in addition that*

$$B_n^3/y_n^{\alpha+1} \to 0 \quad \textit{and} \quad \sup_{k\ge 1} \mathbf{E}X_k^\alpha/\mathbf{E}X_k^2 < \infty$$

for some $\alpha > 2$. Then assertion (10) of Theorem 2 holds with $\sqrt{t_n} = B_n$.

For the proof note first of all that from (7) and (8) we have:

$$\sqrt{t_n}\delta_n = \max_{0\le t\le t_n} |s(t) - \tilde{W}_n(t)| \tag{11}$$

where $\tilde{W}_n(t) := \sqrt{t_n}W_n(t/t_n)$ is a new Wiener process. But it was shown in Theorem 1 from [9] that for each $\alpha > 2$ and every $\varepsilon_n y_n > 0$ it is possible to construct a Wiener process $\tilde{W}_n(\cdot)$ such that

$$\mathbf{P}\left(\max_{t\le B_n^2} |s(t) - \tilde{W}_n(t)| > C\alpha\varepsilon_n y_n\right) \le (\varepsilon_n y_n)^{-\alpha} \sum_{k=1}^{n} \mathbf{E}|X_k|^\alpha,$$

where C is an absolute constant. Hence, Theorem 3 is a partial case of Theorem 2 with $t_n = B_n^2$ and $\varepsilon_n = B^{3/(\alpha+1)}/y_n \to 0$.

Note that if in the proof of Theorem 3 instead of the estimates from [9] we will use the similar estimates from [10] or [11] for $2 < \alpha \le 3$, then we obtain that the main assertion of Theorem 3 remains valid also for the special martingales which was considered in [10] and [11].

3. Conclusions

The process $s(\cdot)$ defined in (7) does not dependent on n. Hence, it follows from (11) that condition (8) holds only when $y_n \to \infty$. It means that strong approximations do not allow as to investigate probabilities from (2) for fixed values y. For such values y we should search for different approaches. And one such approach was developed in [2].

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