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A stopped random walk and stability of a service process of Poisson input flows by a loop algorithm

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Abstract. A queueing system with two Poisson input flows, infinite capacity queues, and a single server is studied. If the first queue is empty at the service time termination of the second queue, the server makes the loop by prolongation of the service time for the second queue by the same amount until the first queue is entered by a customer. Otherwise, a cyclic switching is used. We employ the fact that during certain intervals the second queue is described by a random walk stopped at a random time. In result the queueing system is modeled by a multidimensional discrete Markov chain, the server state and queues' lengths its elements. A necessary condition for the stationary probability distribution existence is found.

Keywords: conflicting queueing system, loop control algorithm, stopped random walk stationary probability distribution, necessary condition.

1. Introduction

In queueing systems with several conflicting input flows different control algorithms are used. Some control algorithms allow for a relatively easy stability study, e.g. cyclic algorithms. On the contrary, control algorithms relying on state-dependent switching of the server can lead to analytically intractable models [1–5]. Stability analysis of queueing systems like these is complicated by varying server regimes. Consider a queueing system with two input flows controlled by an algorithm with a loop. Assume that the server dedicates a nonrandom amount of time to each queue in turn. If the first queue is empty at the service time termination of the second queue, the server makes the loop by prolongation of the service time for the second queue by the same amount until the first queue is entered by a customer. After the first queue no prolongation is possible. A control algorithm of this kind can be used for traffic control at intersections governed by trafficlight signals as well as for automated microchip production machines. Depending on the inputs intensities, the operational metrics of such system are more like those of a purely cyclic system (heavy load case), or an M/G/1 batch system (light first flow case), or a mixture of the two. In turn, the mixing weights depend on all system parameters in quite a complicated way. We aim to demonstrate that a careful choice of observation instants can facilitate the analysis at cost of explicit solution of a stopped random walk problem. The loop algorithm can be regarded as a cyclic algorithm with random durations of server regimes.

2. Main section

Consider a queueing system with two conflicting Poisson inputs Π_1, Π_2. The intensity of Π_j is λ_j, $j = 1, 2$. Customers from Π_j join a queue O_j of unlimited capacity. A server has two states, $\Gamma^{(1)}$ and $\Gamma^{(2)}$. Only customers from O_j get serviced in the state $\Gamma^{(j)}$. The server spends a constant time T_j in the state $\Gamma^{(j)}$. When this time elapses, the server instantly switches to the state $\Gamma^{(2)}$ if $j = 1$, but if $j = 2$ the new server state becomes $\Gamma^{(1)}$ only if the queue O_1 is non-empty, otherwise a new time slot in the state $\Gamma^{(2)}$ takes place. The server loops in $\Gamma^{(2)}$ until new arrivals from Π_1. Epochs of these T_1- and T_2-time endings will be called the control epochs, and denoted τ_i, $i = 0, 1, \ldots$. To define the service process we use the notion of a saturation flow [6]. In the state $\Gamma^{(j)}$ the saturation flow Π_j^{sat} holds $\ell_j > 0$ customers during the time T_j, and the orther saturation flow Π_r^{sat}, $r \neq j$, holds no customers.

An example of a real-life queueing situation satisfying the above assumptions is an intersection with state-dependent traffic light switching. If yellow light signals (when cars may pass) can be adjoined to green light signals, and if a lower-priority direction can be let through only when there are no vehicles in a perpendicular high-priority direction, then our assumptions are fulfilled.

Using methods from [7] we can represent the queueing system as an abstract control system of Lyapunov–Yablonsky [8] and define a multidimensional stochastic sequence

$$\{(\Gamma_i, \varkappa_{1,i}, \varkappa_{2,i}); i = 0, 1, \ldots\} \tag{1}$$

on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, where Γ_i is the server state during he time slot $(\tau_{i-1}, \tau_i]$, $\varkappa_{j,i}$ is the number in O_j at time τ_i, $i = 1, 2, \ldots$, Γ_0 is the initial server state at time $\tau_0 = 0$, dependence of the elementary outcome ω implied but omitted in notation, as usual. We have on the set $\{\omega\colon \Gamma_i = \Gamma^{(2)}, \varkappa_{1,i} > 0\}$ functional relations $\Gamma_{i+1} = \Gamma^{(1)}$, $\varkappa_{1,i+1} = \max\{0, \varkappa_{1,i} + \eta_{1,i} - \ell_1\}$, $\varkappa_{2,i+1} = \varkappa_{2,i} + \eta_{2,i}$, $\eta_{j,i}$ possessing a Poisson probability distribution with parameter $\lambda_j T_1$, $j = 1, 2$, and on the set $\{\omega\colon \Gamma_i = \Gamma^{(1)}\} \cup \{\omega\colon \Gamma_i = \Gamma^{(2)}, \varkappa_{1,i} = 0\}$ we have $\varkappa_{1,i+1} = \varkappa_{1,i} + \eta_{1,i}$, $\varkappa_{2,i+1} = \max\{0, \varkappa_{2,i} + \eta_{2,i} - \ell_2\}$, $\Gamma_{i+1} = \Gamma^{(2)}$, $\eta_{j,i}$ having a Poisson probability distribution with parameter $\lambda_j T_2$, $j = 1, 2$. In effect, sequence (1) is a homogeneous Markov chain.

Let $S' = \{\Gamma^{(1)}, \Gamma^{(2)}\} \times \{0, 1, \ldots\} \times \{0, 1, \ldots\}$ be the state space of the process (1), put $S_0 = \{(\Gamma^{(2)}, 0, x_2); x_2 = 0, 1, \ldots\}$. Introduce stopping moments $\theta_0 = 0$, $\theta_{i+1} = \min\{k\colon k > \theta_i, (\Gamma_k, \varkappa_{1,k}, \varkappa_{2,k}) \notin S_0\}$. Set $\hat{\Gamma}_i = \Gamma_{\theta_i}$, $\hat{\varkappa}_{j,i} = \varkappa_{j,\theta_i}$, $i = 0, 1, \ldots$. The new sequence

$$\{(\hat{\Gamma}_i, \hat{\varkappa}_{1,i}, \hat{\varkappa}_{2,i}); i = 0, 1, \ldots\} \tag{2}$$

is another Markov chain. To find its transition probabilities, let us note that on a set $\{\omega\colon \hat\Gamma_i = \Gamma^{(2)}, \hat\varkappa_{1,i} > 0\} \cup \{\omega\colon \hat\Gamma_i = \Gamma^{(1)}, \hat\varkappa_{1,i} > 0\}$ one has $\theta_{i+1} = \theta_i + 1$, while on a set $\{\omega\colon \hat\Gamma_i = \Gamma^{(1)}, \hat\varkappa_{1,i} = 0\}$ the number of prolongations is $\nu_{i+1} = \theta_{i+1} - \theta_i$, geometrical random variable taking on value $k = 1, 2, \ldots$ with probability $(1-p)p^{k-1}$, $p = e^{-\lambda_1 T_2}$. Also on the set $\{\omega\colon \hat\Gamma_i = \Gamma^{(1)}, \hat\varkappa_{1,i} = 0, \hat\varkappa_{2,i} = x_2\}$ quantities $\varkappa_{2,\theta_i}, \varkappa_{2,\theta_i+1}, \ldots, \varkappa_{2,\theta_{i+1}}$ behave as the number in an $M/G/1/\infty$ queue with batch service and initial queue length $\hat\varkappa_{2,i}$, and $\hat\varkappa_{2,i+1}$ is the number at the stopping time ν_{i+1}. To employ this observation, let us introduce auxiliary i.i.d. Poisson variables η_i', $i = 1, 2, \ldots$ with parameter $\lambda_2 T_2$, and variables $\kappa_0' = b$, $\kappa_{i+1}' = \max\{0, \kappa_i' + \eta_i' - \ell_2\}$. Further, let us introduce a family of probability generating functions ($|z| \leqslant 1$, $k = 0, 1, \ldots$):

$$\Phi_k(z,b) = \mathsf{E}(z^{\kappa_k'}), \quad \Phi(p,z;b) = \sum_{k=0}^{\infty} p^k \Phi_k(z;b), \quad q_j(z;t) = \exp\{\lambda_j t(z-1)\}.$$

Then, in law,

$$\mathsf{E}(z_1^{\hat\kappa_{1,i+1}} z_2^{\hat\kappa_{2,i+1}} \mid \{\omega\colon \hat\Gamma_i = \Gamma^{(1)}, \hat\varkappa_{1,i} = 0, \hat\varkappa_{2,i} = x_2\}) =$$
$$= \sum_{k=1}^{\infty} (e^{-\lambda_1 T_2})^{k-1} \Big(\sum_{b=1}^{\infty} z_1^b \frac{(\lambda_1 T_2)^b)}{b!} e^{-\lambda_1 T_2} \Big) \Phi_k(z_2; x_2) =$$
$$= (e^{\lambda_1 T_2 z_1} - 1)(\Phi(p, z_2; x_2) - z_2^{x_2}).$$

We claim the following:

Lemma 1. *Let $\beta_j = \beta_j(p)$, $j = 1, 2, \ldots, \ell_2$, be the zeroes of an equation $z^{\ell_2} - q_2(z;T_2) = 0$ lying inside a unit disk $|z| < 1$. Then*

$$\Phi(p,z;b) = \frac{(z-\beta_1)\times\ldots\times(z-\beta_{\ell_2})}{z^{\ell_2} - pq_2(z;T_2)} \Bigg(\frac{1}{(1-\beta_1)\times\ldots\times(1-\beta_{\ell_2})} +$$
$$+ \sum_{j=1}^{\ell_2} \frac{(\beta_j)^{\ell_2 - 1}}{\prod\limits_{s\neq j} (\beta_j - \beta_s)} \Big(\frac{z^{b+1} - (\beta_j)^{b+1}}{z - \beta_j} - \frac{1 - (\beta_j)^{b+1}}{1 - \beta_j} \Big) \Bigg).$$

Let $Q_i(r; x_1, x_2) = \mathsf{P}(\{\omega\colon \hat\Gamma_i = \Gamma^{(r)}, \hat\varkappa_{1,i} = x_1, \hat\varkappa_{2,i} = x_2\})$,

$$\Psi_i(z_1, z_2; r) = \sum_{x_1=1}^{\infty} \sum_{x_2=0}^{\infty} z_1^{x_1} z_2^{x_2} Q_i(r; x_1, x_2), \quad r = 1, 2.$$

Lemma 2. *One has* ($p = e^{-\lambda_1 T_2}$)

$$\Psi_{i+1}(z_1, z_2; 1) + \sum_{x_2=0}^{\infty} Q_{i+1}(1; 0, x_2) z_2^{x_2} =$$

$$= q_2(z_2; T_1) \sum_{x_1=1}^{\ell_1-1} \sum_{x_2=0}^{\infty} Q_i(2; x_1, x_2) z_2^{x_2} \sum_{b=0}^{\ell_1-1-x_1} \varphi_1(b; T_1)(1 - z_1^{x_1+b-\ell_1}) +$$

$$+ z_1^{-\ell_1} q_1(z_1; T_1) q_2(z_2; T_1) \Psi_i(z_1, z_2; 2),$$

$$\Psi_{i+1}(z_1, z_2; 2) = q_1(z_1; T_2) z_2^{-\ell_2} q_2(z_2; T_2) \Psi_i(z_1, z_2; 1) +$$

$$+ \sum_{x_1=1}^{\infty} \sum_{x_2=0}^{\ell_2-1} Q_i(1; x_1, x_2) z_1^{x_1} q_1(z_1; T_2) \sum_{b=0}^{\ell_2-1-x_2} \varphi_2(b; T_2)(1 - z_2^{x_2+b-\ell_2}) +$$

$$+ (p^{-1} q_1(z_1; T_2) - 1) \sum_{x_2=0}^{\infty} Q_i(1; 0, x_2) z_2^{x_2} \Big(z_2^{-x_2} \Phi(p, z_2; x_2) - 1 \Big).$$

Lemma 1 is an extension of known facts firstly because as a rule only the case $b = 0$ is studied in the majority of researches. Secondly, they were more interested in the limit of $(1 - p)\Phi(p, z; b)$ as $p \to 1$ as it gives the stationary probability distribution for the random walk.

Theorem 1. *For the existence of a stationary probability distribution of the Markov chain* $\{(\hat{\Gamma}_i, \hat{\varkappa}_{1,i}); i = 0, 1, \dots\}$ *an inequality* $\lambda_1(T_1 + T_2) - \ell_1 \leqslant 0$ *is necessary, and the inequality* $\lambda_1(T_1 + T_2) - \ell_1 < 0$ *is sufficient.*

Theorem 2. *Let* $\alpha_1 = 1, \alpha_2, \dots, \alpha_{l_1}$ *be the zeroes of the equation* $z^{\ell_1} - q_1(z; T_1 + T_2) = 0$ *lying in the unit disk* $|z| \leqslant 1$. *Then the stationary probabilities* $Q_1(1, 0)$ *of the set* $\{(\Gamma^{(1)}, 0, x_2); x_2 = 0, 1, \dots\}$, $Q_1(2, b)$ *of the set* $\{(\Gamma^{(2)}, b, x_2); x_2 = 0, 1, \dots\}$, $b = 2, 3, \dots, \ell_1 - 1$ *are the solution to next linear algebraic system*

$$\sum_{x=1}^{\ell_1-1} Q_1(2, x) \sum_{b=0}^{\ell_1-1-x} \frac{(\lambda_1 T_1)^b}{b!} e^{-\lambda_1 T_1} (\alpha_w^{\ell_1} - \alpha_w^{x+b}) +$$

$$+ Q_1(1, 0) e^{-\lambda_1 T_2} \cdot \frac{e^{\lambda_1(T_1+T_2)(\alpha_w - 1)} - e^{\lambda_1 T_1(\alpha_w - 1)}}{1 - e^{-\lambda_1 T_2}} = 0, \quad w = 2, 3, \dots, \ell_1,$$

$$\sum_{x=1}^{\ell_1-1} Q_1(2, x) \sum_{b=0}^{\ell_1-1-x} \frac{(\lambda_1 T_1)^b}{b!} e^{-\lambda_1 T_1} (\ell_1 - x - b) +$$

$$+ Q_1(1, 0) e^{-\lambda_1 T_2} \frac{\lambda_1 T_2}{1 - e^{-\lambda_1 T_2}} = \frac{\ell_1 - \lambda_1(T_1 + T_2)}{2}.$$

Theorem 3. *For the existence of the stationary probability distribution of the Markov chain* (2) *it is necessary that*

$$\frac{1}{2}(\lambda_2(T_1+T_2)-\ell_2)+\frac{p}{1-p}(\lambda_2 T_2-\ell_1)Q_1(1,0)<0.$$

Theorems 1–3 characterize the stability regime of the queueing system with loop algorithm in a form suitable for numerical verification.

3. Conclusions

We have demonstrated that a queueing system governed by a loop algorithm can be efficiently studied if the embedded process resembles a cyclic algorithm. It can be achieved by skipping a random number of working tacts in certain server regimes.

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