UDC 519.21

Moments of the sojourn time of random walk above a certain boundary

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Abstract. We study the behaviour of moments of the sojourn time of random walk with zero drift above a certain growing boundary. With imposed Cramer condition on the existence of an exponential moment on the distribution of jumps of the random walk we find asymptotic expansions for the expectation of the sojourn time.

Under general conditions, we obtain inequalities for certain moments of sojourn time of a random walk over linear boundary. We find asymptotics of these moments for random walks with regular or semi-exponential distribution of summands.

Keywords: random walk, sojourn time, asymptotic analysis, inequalities.

1. Introduction

Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables, $S_n := \sum_{i=1}^{n} \xi_i$. Sojourn time of random walk $\{S_n\}_{n\geqslant 1}$ above a level b is defined as

$$T_n(b) := \sum_{i=1}^{n} I_{\{S_i \geqslant b\}},$$

where I_A — is an indicator of event A. The value of b can also be dependent on n.

The study of distribution of sojourn time on half–axis is a challenging task. For the simple random walks a set of results about it's distribution can be obtained via combinatorial methods: arcsine law is a well known example, that describes limit behaviour of distribution of $T_n(0)$. Other results are based on convergence of functionals of trajectory of random walk to distribution of corresponding functionals of limit processes. Here we will focus on a more specific problem of studying asymptotics of moments of $T_n(b)$.

2. Main section

Denote the rate function

$$\Lambda(\alpha) := \sup_{\lambda\in\mathbb{R}}(\alpha\lambda - \ln\psi(\lambda)),$$

where $\psi(\lambda) := \mathbb{E}e^{\lambda\xi}$.

Following simple result describes asymptotic of $\mathbb{E}T_n(b)$ under the Cramer condition:

Theorem 1. *Assume $\mathbb{E}\xi = 0$ and $|\mathbb{E}e^{\lambda\xi}| < \infty$, given that $|\mathrm{Re}\lambda| \leqslant \beta$ for some $\beta > 0$. If b_n — monotone sequence such that, $b_n/\sqrt{n} \to \infty$ and $b_n/n \to 0$ as $n \to \infty$, then*

$$\mathbb{E}T_n(b_n) \underset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi}} \frac{n^{5/2}\sigma^3}{b_n^3} e^{-n\Lambda(b_n/n)}.$$

Needs to be noted that the study of sojourn time is certainly related to boundary value problems for random walks. And, similarly to other papers devoted to the study of boundary value problems, the asymptotic analysis of expectation of sojourn time under the Cramer conditions can be carried out using the so-called factorization method, which allows us to additionally obtain a complete asymptotic expansion for $\mathbb{E}T_n(b_n)$.

Next, all obtained inequalities for moments of sojourn time are based on the following result:

Theorem 2. *Let $n \geqslant 1$ and $g(x)$ — be such a function, that $g(0) = 0$ and*

$$\frac{d^k g}{dx^k}(x) \geqslant 0$$

for all $x \in (0, n)$ and all natural $k \leqslant n$. Then

$$\mathbb{E}g(T_n(b)) \leqslant \sum_{j=1}^{n} (g(n-j+1) - g(n-j))\, \mathbb{P}\{S_j \geqslant b\},$$

$$\mathbb{E}g(T_n(b)) \geqslant \sum_{j=1}^{n} (\mathbb{E}g(\eta_{n-j}+1) - \mathbb{E}g(\eta_{n-j}))\, \mathbb{P}\{S_j \geqslant b\},$$

where random variables $\{\eta_i\}_{i=1}^{n}$ have binomial distribution with parameters i and $p := \min_{1\leqslant j\leqslant n} \mathbb{P}\{S_j \geqslant 0\}$.

In particular, for $k \in \mathbb{N}$ and $\mu > 0$ we get:

$$\mathbb{E}T_n^k(b) \geqslant \sum_{j=1}^{n} \left((p(n-j)+1)^k - (p(n-j))^k\right) \mathbb{P}\{S_j \geqslant b\},$$

$$\mathbb{E}T_n^k(b) \leqslant \sum_{j=1}^{n} \left((n-j+1)^k - (n-j)^k\right) \mathbb{P}\{S_j \geqslant b\},$$

$$1+(e^\mu-1)\sum_{j=1}^{n} q_1^{n-j}\mathbb{P}\{S_j \geqslant b\} \leqslant \mathbb{E}e^{\mu T_n(b)} \leqslant 1+(e^\mu-1)\sum_{j=1}^{n} q_2^{n-j}\mathbb{P}\{S_j \geqslant b\},$$

where $q_1 = (1 - p + pe^{\mu})$, $q_2 = e^{\mu}$.

In some cases the upper bound will be asymptotically equivalent to $\mathbb{E}g(T_n)$. For example, following result holds:

Theorem 3. *Let* $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 < \infty$, $b_n/\sqrt{n} \underset{n\to\infty}{\longrightarrow} \infty$ *and assume that exists such sequence of positive numbers* Δ_n, *that*

$$\Delta_n/\sqrt{n} \underset{n\to\infty}{\longrightarrow} \infty, \quad \mathbb{P}\{S_j \geqslant b_n + \Delta_n\} \underset{n\to\infty}{\sim} \mathbb{P}\{S_j \geqslant b_n\},$$

uniformly for $1 \leqslant j \leqslant n$.

Then for any monotone function g, *such that* $g(0) = 0$, *following holds:*

$$\mathbb{E}g(T_n(b_n)) \underset{n\to\infty}{\sim} \sum_{j=1}^{n} (g(n-j+1) - g(n-j))\, \mathbb{P}\{S_j \geqslant b_n\}.$$

It can be shown that regularly varying distributions satisfy conditions of this theorem.

Acknowledgments

The work is partially supported by RFBR grant No 16-00-00049.