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Simulation of Branching Random Walks on Multidimensional Lattices

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Abstract. We present an approach to simulation of continuous-time branching random walks on multidimensional lattices with a finite number of particle generation centers of different types located at the lattice points. Such processes introduced by Yarovaya (2012) can be used for modelling of a particle population dynamics. For example, exponential growth of the particle population in the frame of branching random walks models may be explained by the excess of the threshold value. Simulation of branching random walks is applied for numerical estimation of a threshold value of the parameter on limited time intervals. Simulation of the process is based on a well-known algorithm of queue data structures and estimates obtained by the Monte-Carlo method.

Keywords: branching random walks, simulation, Monte-Carlo method, queue.

1. Introduction

We consider a continuous-time branching random walks (BRWs) on a multidimensional lattice $\mathbf{Z}^d$, $d \geqslant 1$, with a finite number of sources located at the lattice points x_i. We assume that at the initial moment there is a single particle which walks on the lattice until it reaches one of the sources where its behavior changes. In source particle can die, leave $n > 1$ newborn particles or walk.

The random walk is specified by an infinite matrix $A = (a(x,y))_{x,y\in\mathbf{Z}^d}$ of transition intensities: $a(x,y) \geq 0$ for $x \neq y$, $a(x,x) < 0$; $a(x,y) = a(y,x) = a(0,y-x) = a(y-x)$ and $\sum_z a(z) = 0$. Each of a few sources located at the some point x_i is specified by the infinitesimal generation function $f_{x_i}(u) := \sum_{n=0}^{\infty} b_{x_i,n}u^n$, where $b_{x_i,n} \geq 0$ for $n \neq 1$, $b_{x_i,1} < 0$ and $\sum_n b_{x_i,n} = 0$. We assume that $\beta_{x_i,r} := f_{x_i}^{(r)}(1) < \infty$, $r \in \mathbf{N}$, and denote $\beta_{x_i} := \beta_{x_i,1}$.

The evolution of a particle in such a BRW is performed in accordance with the following rules: being outside of the set $\{x_i\}$, say at a point x, the particle performs a random walk: the particle stays at the point x a random time distributed in accordance to the exponential law with parameter $-a(0)$ and then jumps to a point $y \neq x$ with the probability $-a(y-x)/a(0)$; being at a point $x \in \{x_i\}$, say at a point $x = x_i$, the particle performs a symmetric random walk or branching. The branching in this case is performed in accordance with the Bienayme-Galton-Watson process specified by the infinitesimal generation function $f_{x_i}(u)$. So, in this case the behavior of the particle is assumed as follows: first it stays at the point x a random time distributed in accordance to the exponential law with parameter $-(a(0)+b_{x_i,1})$ and then either jumps to a point $y \neq x$ with the probability $-a(y-x)(a(0)+b_{x_i,1})$ or generates $n \neq 1$ offsprings with the probability $-b_{x_i,n}/(a(0)+b_{x_i,1})$. Notice that in the case $n = 0$ the particle dies.

All particles will behave the same way independently of each other and of their history, for further details, see i.e. [1].

The main object of study is the evolution operator for the local mean number $m_1(t,x,y)$ at an arbitrary point $y \in \mathbf{Z}^d$ and total mean number on the entire lattice $m_1(t,x) = \sum_{y \in \mathbf{Z}^d} m_1(t,x,y)$ and their moments, where x is coordinate of initial particle. Supercritical behavior type, unlike other types(critical or subcritical) is characterized by the exponential growth of the mean number of particles $m_1(t,x,y)$ and $m_1(t)$ and this distinction often play a central role in practice. It was shown that mean number moments satisfy linear differential operator equations in Banach spaces and general analysis of this operator was done in [2]. For instance, as a result, in case of BRWs with source with equal intensity β, this value β determine asymptotic behavior of mean number of particles $m_1(t,x,y)$ and $m_1(t)$: threshold value β_{cr} separates supercritical, critical or subcritical behavior types. For the supercritical BRWs exponential growth of the number of amounts $m_1(t,x,y)$ and $m_1(t)$ is evoked with the presence of a positive eigenvalue in the spectrum of the corresponding evolution operator.

Mean number moment equations are difficult differential equations and can't be solved explicitly. In practice mean number moments and other characteristic of the BRWs may be obtained by utilizing numerical methods, like Monte Carlo (see [5]). Monte Carlo techniques uses a large number of process simulations, to estimate different characteristic of the model including. Simulation algorithms were used in a number of works (see, e.g., [7–10]), in that random walks or branching processes were studied about separately. However, we didn't find papers uses Monte Carlo method for studying BRWs model.

2. Main section

Description of BRWs model given in the previous section is well suited for algorithm design. We introduced an algorithm for modelling BRWs based on queue data structure (for details, see i. e. [6]) and show techniques to estimate such BRWs characteristics, like moments of the local mean number $m_n(t,x,y)$ and moments of the total mean number $m_n(t)$ and threshold value β_{cr}.

We propose method of simulation BRWs with several sources on finite time intervals. The property of BRWs that we based on in constructing simulation algorithm is the stochastically independence of particles behavior. This allows to simulate evolution of each particle of BRWs on a finite time interval separately and independently from others.

Algorithm simulates evolution of the one selected particle in a given time interval $[0, T]$. During this process new particles can appear, when the selected particle is branching in one of the sources. Information about such events (coordinate and birth time of newborn particle (x, t)) is added in the queue. When the evolution of the selected particle in the given time interval $[0, T]$ is simulated, algorithm extract next particle from the queue. The algorithm terminates when queue is empty, i.e. evolution of initial particle and evolution of all their descendants was simulated.

Initialization. First, we choose the finite sets of points $\{x_i\}_{i=1}^k$ and specify all the necessary quantities determining the matrix of transition intensities A, and the infinitesimal generating functions $f_{x_i}(u)$ with $i = 1, 2, \ldots, k$ and $\bar{f}_{x_i}(u)$ with $i = 1, 2, \ldots, n$. We also specify the length of the interval $[0, T]$ on which simulation will be conducted. At last, we fix an initial point x_0 and added information about initial particle $(x_0, 0)$ to queue.

Loop. We extract from the queue first pair (x, t) corresponding to newborn particle in x at the moment t and simulate evolution of this particle during time interval $[t, T]$, repeating the simulation of particle evolution while $t < T$ according rules described above.

We calculate sojourn time Δt accordance with the placement of the point x, pass to the moment $t = t + \Delta t$ and simulate evolution at this moment using random number generators. If particle performed branching and $n > 1$ particles appeared at the point x, we continue to simulate evolution of one of this n particles from time t and add remaining $n - 1$ pairs (x, t) to queue. If $n = 0$ particle dies and this step is finished. In case of walking particle jumps to a point $y \neq x$ and we continue simulation with $x = y$.

Termination. Loop repeats while the queue is not empty.

Collecting of data. After the termination of the algorithm, following to the Monte-Carlo method, we repeat simulation with the same parameters (but with different runs of random number generators) several times to collect the needed number of data samples (simulations) which would be sufficient for statistical data treatment. After collecting all the data we start the evaluation of the characteristics of the BRW under consideration.

As is clear from the description above, the algorithm is naturally randomized. When we initialize random number generator with different values we obtain different realizations of BRWs. Examples of using the algorithm to obtain moments $m_n(t, x, y)$, $m_n(t)$ and threshold value β_{cr} are described below.

We consider that the number of particles at point and the number of particles on the lattice does not vary significantly during small time intervals. We calculated the number of particles at the time instants $t_i = \frac{i}{N}T,\ i = 0, \ldots, N$ as sum of the number of particles at t_{I-1} and increment of the number of particles on interval $[t_i, t_{i+1}]$. At moment $t_0 = 0$ local number of particles at the point and total number on the lattice is determined according to the initial conditions. The increment on time interval updates directly when the evolution of particle is simulated (increments when new particle borns or decrements when particle dies). Sampling mean numbers and their moments at the time instants $t_i = \frac{i}{N}T,\ i = 0, \ldots, N$ are computed in accordance to the definition using samples of total and local particle numbers, obtained at different values initialising random number generators.

Hereafter, we'll consider threshold value β_{cr} for BRWs with one source or with a few sources of equal intensity. As is known exponential asymptotic of particle number both at the arbitrary point and on the entire lattice is conditioned by source intensity β exceeding threshold value β_{cr}. In case $\beta < \beta_{cr}$ exponential growth is not observed.

Learning to classify the process as exponential or not exponential (supercritical and subcritical respectively), we can estimate threshold using binary search (for details see i. e. [6]). We start a search on the interval $[L, R],\ L < \beta_{cr} < R$. If BRWs with source intensity $\beta = M = \frac{L+R}{2}$ is supercritical (total mean number is classified as exponentially growing function) then we will continue search on interval l $[L', R'] = [L, M]$, else if BRWs is subcritical then continue search on interval $[L', R'] = [M, R]$. Obtained interval $[L', R']$ contains β_{cr}, so we repeat the same procedure on $[L', R']$. Interval length $|R' - L'| = \frac{1}{2}|R - L|$ decreases by half every time. It may be difficult to classify process as exponential or not exponential when β is close to β_{cr}, so in that case search must be terminated or continue on a longer time interval. If $|R - L| < \varepsilon$, where ε is required accuracy search also terminates.

3. Conclusions

The exhaustive classification of the limit behavior (up to a scalar factor) of the total mean number of particles and local mean number of particles at the source for BRWs with one source is given in [1]. Obtained total and local mean numbers of BRWs with one source demonstrates that exponential asymptotic of subcritical processes as well as descending asymptotic can be observed on finite intervals.

Note that described algorithm is not suitable for modelling BRWs with a truly infinite matrix of transition intensities A or infinitesimal generating functions $f_{x_i}(u)$.

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References

1. *Yarovaya, E.B.* Spectral properties of evolutionary operators in branching random walk models // Mathematical Notes. — 2012. — Vol. 92, no. 1. — P. 115–131.
2. *Yarovaya, E.B.* Branching random walks with several sources // Mathematical Population Studies. — 2013. — Vol. 20, no. 1. — P. 14–26.
3. *Yarovaya, E.B.* Branching random walks in a heterogeneous environment. — Center of Applied Investigations of the Faculty of Mechanics and Mathematics of the Moscow State University, Moscow. 2007. (In Russian)
4. *Yarovaya, E.B.* The structure of the positive Discrete spectrum of the evolution operator arising in branching random walks // Doklady Mathematics. — 2015. — Vol. 92, no. 1. — P. 507–510. — DOI 10.1134/S1064562415040316.
5. *Fishman, George S.* Monte Carlo: concepts, algorithms, and applications. — Springer.— 1996. — ISBN 0-387-94527-X.
6. *Cormen, T.H., Leiserson, C.E., Rivest, R.L., Stein, C.* Introduction to algorithms, third edn. — MIT Press, Cambridge, MA, 2009.
7. *Andries E., Umarov S., Steinberg St.* Monte Carlo Random Walk Simulations Based on Distributed Order Differential Equations with Applications to Cell Biology // Fractional Calculus and Applied Analysis. — 2006. — Vol. 9, no. 4. — P. 351–369.
8. *Kleinhans D., Friedrich R.* Continuous Time Random Walks (CTRWs): Simulation of continuous trajectories // Phys. Rev. E. — 2007. — Vol. 76, 061102 — DOI 10.1103/PhysRevE.76.061102.
9. *Fabricio Murai, Bruno Ribeiro, Don Towsley* Characterizing Branching Processes from Sampled Data / Proceedings of the 22Nd International Conference on World Wide Web, WWW '13 Companion, Rio de Janeiro, Brazil, 2013. — P.805–812. — DOI10.1145/2487788.2488053
10. *Daskalova N.* EM Algorithm for Estimation of the Offspring Probabilities in Some Branching Models // Mladenov V.M., Ivanov P.C. (eds) Nonlinear Dynamics of Electronic Systems. NDES 2014 / Communications in Computer and Information Science, Vol. 438. — Springer, Cham., 2014. — DOI 10.1007/978-3-319-08672-9_23