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Survival Analysis and Recurrence Criteria for Branching Random Walks

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Abstract. The models of symmetric continuous-time branching random walks on a multidimensional lattice with a few sources of particle birth and death are studied. Emphasis is made on the survival analysis and study of branching random walk properties depending on the configuration of the sources and their intensities. In particular, we will try to describe how the properties of a branching random walk depend on such characteristics of an underlying branching walk as finiteness or infiniteness of the variance of jumps. The presented results are based on Green's functions representations of transition probabilities of a branching random walk.

Keywords: branching random walks, recurrence criteria, Green's function.

1. Introduction

It is a common practice to describe branching random walks (BRWs) in terms of birth, death and walk of particles, which makes it easier to use them in various areas of nature sciences, see, e.g., [1, 2] and references therein. Application in the reliability theory of BRWs on multidimensional lattices with one centre of particle generation and finite variance of walk jumps was discussed in [3]. Emphasis in the present work is made on the survival analysis and study of BRW properties depending of the configuration of the sources and their intensities. The answer to these and other questions heavily depends on numerous factors which affect the properties of a BRW. Therefore, we will try to describe how the properties of a BRW depend on such characteristics of an underlying branching walk as finiteness or infiniteness of the variance of jumps. The presented results are based on Green's functions of transition probabilities of an underlying branching walk.

The description of the model of a symmetric BRW with a finite number of branching sources is given in Sec. 2, where we formulate also the recurrence criteria for BRWs in terms of Green's functions and recall some recent results on the symmetric BRWs. In Sec. 3 we recall some properties of the local extension probability of the process at the origin and of survival probability of the particle population without any assumptions on the variance of jumps of an underlying branching walk In Sec. 4 we consider an operator model of BRWs with the branching sources of equal intensities and, in particular, the case in which the local extension probabilities at every lattice point tend to 1, as $t \to \infty$. The main results are formulated in Sec. 4.

2. Model and Previous Results

Evolution of the particle system on $\mathbf{Z}^d$ is described by the number of particles at time t at each point $y \in \mathbf{Z}^d$ on the assumption that at the time $t = 0$ the system consists of one particle located at the point x. The particle walks on $\mathbf{Z}^d$ until it reaches one of the points $x_1, x_2, \ldots, x_N$, $N < \infty$, called *branching sources*, where it can die or produce a random number of offsprings. It is assumed that evolution of the newborn particles obeys the same law independently of the rest of the particles and the prehistory. Now we proceed to a full description of the model.

The random walk of particles is defined by the infinitesimal transition matrix $A = \|a(x,y)\|_{x,y\in\mathbf{Z}^d}$ and assumed to be symmetric, $a(x,y) = a(y,x)$, homogeneous, $a(x,y) = a(0, y-x) = a(y-x)$, irreducible, that is, for every $z \in \mathbf{Z}^d$ there exists a set of vectors $z_1, z_2, \ldots, z_k \in \mathbf{Z}^d$ such that $z = \sum_{i=1}^{k} z_i$ and $a(z_i) \neq 0$ for $i = 1, 2, \ldots, k$, regular, that is $\sum_{x\in\mathbf{Z}^d} a(x) = 0$, where $a(x) \geq 0$ for $x \neq 0$ and $a(0) < 0$.

The reproduction law at each source x_i is defined by the continuous-time Bienaymé-Galton-Watson branching processes with the infinitesimal generation function $f(u, x_i) = \sum_{n=0}^{\infty} b_n(x_i)u^n$, $0 \leq u \leq 1$, where $b_n(x_i) \geq 0$ for $n \neq 1$, $b_1(x_i) < 0$ and $\sum_n b_n(x_i) = 0$. We assume $f^{(r)}(1, x_i) < \infty$ for every $r \in \mathbf{N}$. For the investigation the values $\beta_i = f'(1, x_i)$, called the *intensities* of the branching source x_i, will play an important role.

By $p(t, x, y)$ we denote the transition probability of the underlying random walk. This function is implicitly determined by the transition intensities $a(x, y)$ (see, for example, [4]). Then, Green's function of the operator A can be represented as $G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t,x,y)\,dt$, where $\lambda \geq 0$.

The analysis of BRWs depends on whether the value of $G_0 = G_0(0,0)$ is finite or infinite. As is known, a random walk is *transient* if $G_0(0,0) < \infty$ and *recurrent* if $G_0(0,0) = \infty$. We generalize this definition by calling BRW *transient* if the underlying random walk is *transient* and *recurrent* if the underlying random walk is *recurrent*.

If the variance of jumps of the underlying random walk is finite, i.e.

$$\sum_{z\in\mathbf{Z}^d} |z|^2 a(z) < \infty, \tag{1}$$

where $|z|$ is the Euclidian norm of the vector z, then we get the following *recurrence criteria* for BRWs with *finite variance* of jumps: $G_0 = \infty$ for $d = 1, 2$, and $G_0 < \infty$ for $d \geq 3$, see, e.g. [4]). If

$$a(z) \sim H\left(z/|z|\right)|z|^{-(d+\alpha)}, \quad \alpha \in (0,2), \tag{2}$$

for all $z \in \mathbf{Z}^d$ with sufficiently large norm, where $H(\cdot)$ is a continuous positive function symmetric on $\mathbf{S}^{d-1} = \{z \in \mathbf{R}^d : |z| = 1\}$, then unlike

(1) the series $a(z)$ diverges, which leads to the infinity of the variance of jumps. In this case we get the following *recurrence criteria* for BRWs with *infinite variance* of jumps: $G_0 = \infty$ for $d = 1$ and $\alpha \in [1, 2)$ and G_0 is finite if $d = 1$ and $\alpha \in (0, 1)$ or $d \geq 2$ and $\alpha \in (0, 2)$ resulted from the behavior of $p(t, x, y) \sim h_{d,\alpha} t^{-d/\alpha}$, where $h_{d,\alpha}$ is a positive constant depending on dimension of $\mathbf{Z}^d$ and $t \to \infty$, see details in [5, 6].

The moments of numbers, both at an arbitrary lattice point $\mu_t(y)$ and on the entire lattice $\mu_t = \sum_{y' \in \mathbf{Z^d}} \mu_t(y)$, are denoted by $m_n(t, x, y) := E_x \mu_t^n(y)$ and $m_n(t, x) := E_x \mu_t^n$, $(n \in \mathbf{N})$, respectively, where E_x is the conditional expectation, provided that $\mu_0(\cdot) = \delta_x(\cdot)$. The asymptotic behavior of moments for $N = 1$ under the condition (1) was studied, e.g., in [4].

Recall known results, see, e.g. [4], for a BRW with one branching source that will be needed in Sec. 4. If $\beta > G_0^{-1}$, then the equation

$$G_\lambda(0, 0) = 1 \tag{3}$$

has a single positive solution λ_0, whence it follows that the random variables $\mu_t(y)$ and μ_t have a limit distribution for $t \to \infty$ under the normalization $e^{-\lambda_0 t}$.

For $\beta \leq G_0^{-1}$, the growth of the moments μ_t and $\mu_t(y)$ of the particle numbers appears to be irregular with respect to the moment number n which means that the behavior of the particle numbers μ_t and $\mu_t(y)$, as $t \to \infty$, differs appreciably from the behavior of their moments, and the study of BRW survival probabilities becomes relevant.

3. Survival probabilities

Recall some results on the symmetric BRW which may find use in the studies of reliability. Let us consider for simplicity a BRW with one branching source located at the origin. The notion of the system reliability is related with the notions of the *local extension probability of the process* $Q(t, x, y) = P_x\{\mu_t(y) > 0\}$ and the *survival probability of the particle population* $Q(t, x) = P_x\{\mu_t > 0\}$. The function $Q(t, x)$ may be called the system reliability function. The asymptotic behavior of $Q(t, x, 0)$ describes the number of working elements. That is why consideration was given to the asymptotic behavior of the survival probability $Q(t, x)$ of the particle population on the lattice and the probability $Q(t, x, 0)$ of particle availability at the source at time t. The integral equations for $Q(t, x, y)$ and $Q(t, x)$ were established in [7]. The main result there is that under the condition (2) the equations for $Q(t, x, y)$ and $Q(t, x)$ have the same representation as under the condition (1). But in virtue of the different recurrence criteria for BRWs under the conditions (2) and (1) we obtain another limit theorems in contrast to the theorems established in [3].

4. Supercritical BRWs with Recessing Sources

The moments m_1 satisfy to the evolution equations $\frac{\partial m_1}{\partial t} = \mathcal{H}_\beta m_1$ with the initial conditions $m_1(0,x,y) = \delta_y(x)$, $m_1(0,x) \equiv 1$, respectively, see [8]. In the BRW models with finitely many sources of equal intensity, as was shown in [8], there arise multipoint perturbations of the symmetric random walk generator $\mathcal{A}$ which have the form

$$\mathcal{H}_\beta = \mathcal{A} + \beta \sum_{i=1}^{N} \Delta_{x_i}, \tag{4}$$

where $x_i \in \mathbf{Z}^d$. Here $\mathcal{A} : l^p(\mathbf{Z}^d) \to l^p(\mathbf{Z}^d)$, $p \in [1,\infty]$, is a symmetric operator acting by formula $(\mathcal{A}u)(z) := \sum_{z' \in \mathbf{Z}^\mathbf{d}} a(z-z')u(z')$, and $\Delta_x = \delta_x \delta_x^T$, where $\delta_x = \delta_x(\cdot)$, denotes the column vector on the lattice which is equal to 1 at the point x and to 0 at other points. The summand $\beta \sum_{i=1}^N \Delta_{x_i}$ in (4) can result in the appearance of positive eigenvalues of the operator $\mathcal{H}_\beta$ the number of which (counting their multiplicity) does not exceed *the number of terms* in the sum.

Let β_c be the minimal value of the source intensity such that for $\beta > \beta_c$ the spectrum of $\mathcal{H}_\beta$ has positive eigenvalues. As was proved in [9], if $G_0 = \infty$ then $\beta_c = 0$ for $N \geq 1$ and if $G_0 < \infty$ then $\beta_c = G_0^{-1}$ for $N = 1$, and $0 < \beta_c < G_0^{-1}$ for $N \geq 2$. In [9] it was also shown that in the case $N \geq 2$ the operator $\mathcal{H}_\beta$ with $\beta > \beta_c$ can have at most N positive eigenvalues $\lambda_0(\beta) > \lambda_1(\beta) \geq \cdots \geq \lambda_{N-1}(\beta) > 0$, counting their multiplicity, where the eigenvalue $\lambda_0(\beta)$ has multiplicity one. Moreover, there exists a value $\beta_{c_1} > \beta_c$ such that, for $\beta \in (\beta_c, \beta_{c_1})$ the operator has no other eigenvalues except $\lambda_0(\beta)$. In general, the problem of finding the eigenvalues of an operator is complicated.

In [10] it was assumed that branching sources of equal intensities β are situated in vertices $x_1, x_2, \ldots, x_N$, $N = 2, \ldots, d$, of the regular $(N-1)$-simplex on $\mathbf{Z}^d$ and proved the theorem about the effect of "limiting coalescence" of eigenvalues for the simplex configurations of branching sources.

In the general case, let us consider branching sources of equal intensities β situated at arbitrary points $x_1, x_2, \ldots, x_N$, where $N = 2, \ldots, d$. Denote, for each configuration $S = \{x_1, x_2, \ldots, x_N\}$, by $0 \leq \beta_c(N,S) < \beta_{c_1}(N,S) \leq \cdots \leq \beta_{c_{N-1}}(N,S)$ the critical values of β such that, when β increasing traverses each $\beta_{c_i}(N,S)$, a new positive eigenvalue of $\mathcal{H}_\beta$ bifurcates from zero. Previously considered simplex configurations of the points $x_1, x_2, \ldots, x_N$ in [10] provides an example, in which $\beta_{c_1} = \beta_{c_2} = \cdots = \beta_{c_N}$. Contrary to the simplex configuration case, the values of $\beta_{c_1}(N,S), \ldots, \beta_{c_{N-1}}(N,S)$ may indeed differ from each other.

Theorem 1 *Given $N \geq 2$, a configuration $S = \{x_1, x_2, \ldots, x_N\}$ and $\beta > G_0^{-1}$, then for every positive eigenvalue $\lambda_i(S,\beta,N)$, $i = 0,1,\ldots,N-1$,* *of the operator* $\mathcal{H}_\beta$ *we get* $\lim_{\rho(S)\to\infty} \lambda_i(S,\beta,N) = \lambda_*(\beta)$*, where* λ_* *is the solution of* (3)*.*

Moreover, $0 \le \beta_c(N,S) \le G_0^{-1} \le \beta_{c_1}(N,S) \le \cdots \le \beta_{c_{N-1}}(N,S)$*, and for every* N *we have* $\lim_{\rho(S)\to\infty} \beta_c(N,S) = \lim_{\rho(S)\to\infty} \beta_{c_i}(N,S) = G_0^{-1}$*, where* $\rho(S) := \min_{i\neq j} |x_i - x_j|$*.*

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