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Non-Markovian Models of Branching Random Walks

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Abstract. For applications in reliability theory it is important to reduce dynamical systems with denumerable number of states to systems with a finite number of states. We demonstrate that in such procedure Markovian property of initial dynamical systems with denumerable number of states may be lost. We consider branching processes with transport of particles called branching random walks. Non-Markovian models are constructed based on branching random walks on multidimensional lattices with a few branching sources. The object of investigation in branching random walks is the number of particles in every lattice point. By aggregation of some lattice point areas it is possible to consider a finite set of system states instead of a denumerable set of system states, but, as a result, the Markovian property of initial branching random walks is lost. In such models we assume that the distribution of the sojourn time of the particle at every lattice point is exponential and the underlying random walk is the Markov process on a countable phase space. If we consider the random walk with only two states: in the branching sources and out of them then in this case the random walk becomes Non-Markovian. The results can be used for study of particle population in branching random walk in sources and out of them. General methods are proposed to study non-Markovian models for branching random walks.

Keywords: branching random walks, non-Markovian processes, Green's function.

1. Introduction

For applications, e.g., in the reliability theory, it is important to reduce consideration of dynamical systems with denumerable number of states to consideration of systems with a finite number of states [1].

In the paper we demonstrate that in the course of such a reduction the Markovian property of initial dynamical systems with denumerable number of states may be lost. Non-Markovian models are constructed based on continuous-time branching random walks (BRWs) on $\mathbf{Z}^d$, $d \geq 1$, with a few branching sources, see the BRW description, e.g., in [2]. By aggregating some lattice point areas it is possible to consider a finite set of system states instead of a denumerable set of system states, but, as a result, the Markovian property of initial BRWs is lost. Such approach to study of symmetric and symmetrizable BRWs with one particle generation centre was suggested in [3].

In Sec. 2 we construct a continuous-time symmetric random walk X_t on $\mathbf{Z}^d$, which can be considered as moving of an initial particle on $\mathbb{Z}^d$ without branching with two possible space states at time t. In Sec. 3

we shortly discuss Non-Markovian BRW models and formulate some limit results. Remark that the equations of Sec. 2 may be generalized on X_t with infinite variance of jumps, but the distribution time in the space states, based on the asymptotic behavior transition probabilities of X_t, will be different.

2. Non-Markovian Model of a Random Walk

We assume that the random walk of particles is defined by the infinitesimal transition matrix $A = \|a(x,y)\|_{x,y\in\mathbb{Z}^d}$ which is symmetric: $a(x,y) = a(y,x)$; homogeneous: $a(x,y) = a(0, y-x) = a(y-x)$; irreducible: $\forall\ x \in \mathbb{Z}^d\ \exists\ x_1,\ldots,x_k \in \mathbb{Z}^d : x = \sum_{i=1}^{k} x_i$ and $a(x_i) \neq 0$ for $i = 1,2,\ldots,k$; regular: $\sum_{x\in\mathbb{Z}^d} a(x) = 0$, for $a(x) \geq 0$, $x \neq 0$, $a(0) < 0$, and also has a finite variance of jumps: $\sum_{x\in\mathbb{Z}^d} |x|^2 a(x) < \infty$.

Let us consider a continuous-time symmetric irreducible random walk X_t on $\mathbf{Z}^d$ with two possible space states at time t.

— State I: $X_t = x_i$, $i = 1,2,\ldots,N$, (black lattice points on Fig. 1);

— State II: $X_t = y$, $y \neq \{x_i\}_{i=1}^{N}$, (white lattice points on Fig. 1).

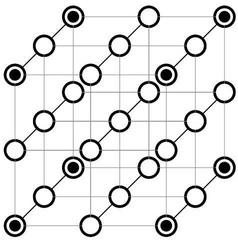

Figure 1. Example of the model on $\mathbb{Z}^3$.

In [3] there was considered the case when $N = 1$ and the random walk started from the origin at time $t = 0$. In this situation the time spent by the particle at the origin until leaving it was denoted by τ_1, and the time spent by the particle outside the origin until its first return to the origin was denoted by τ_2. In virtue of Markovian character of the initial random walk [3] in this case the time τ_1 spent by the particle at the points x_1 was exponentially distributed random variable with parameter $-a(0)$, but the sojourn time τ_2 of X_t to be outside the origin before the first return to the origin was not exponentially distributed and depend on the lattice dimension d.

Now we consider the case $N \geq 2$. Let us assume that X_t resides in State I until its first transition to State II. Due to independence of random variables of time at the points x_i, $i = 1, 2, \ldots, N$, on $\mathbb{Z}^d$ and their identical distributions we can conclude that for every n, $n \le N$, the time τ_1 is defined by the sum of n independent exponentially distributed random variables, each of which has the mean value of $1/(-a(0)$, and has the Erlang distribution with the parameters $(-1/a(0), n)$.

Set $G_1(t) := P(\tau_1 < t)$ and $G_2(t) := P(\tau_2 < t)$. Below, we will obtain that the distribution of the random walk sojourn time τ_2 to be outside the State I on $\mathbb{Z}^d$ (before the first return to State I) is not exponentially distributed. The long-time asymptotic behavior of the transition probability $p(t, x, y)$ defined by the intensities $a(x)$, as was shown, e.g., in [4], is given by $p(t, x, y) \sim \gamma_d t^{-d/2}$, where d is a constant defined by the dimension d of the lattice. Denote by $G_\lambda(x, y) := \int_0^\infty e^{-\lambda u} p(u, x, y) du$ the Laplace transform of transitional probability called *Green's function*. By a standard way it is possible to get the following lemma.

Lemma 1. *For the random walk under consideration we have*

$$p(t, 0, x) = 1 - G_1 + \int_0^t p(t - u, 0, x)(G_1 * G_2)\{du\},$$

where $(G_1 * G_2)(u) = P(\tau_1 + \tau_2 < u)$ *and* $x \in \mathbf{Z}^d$.

From here, applying the Laplace transform and the Tauberian theorems [5] we get the main proposition.

Theorem 1. *Let* $N \ge 1$ *then for the random walk under consideration, as* $t \to \infty$, *we have*

$$\begin{aligned} 1 - G_2(t) &\sim N/(-a(0)\gamma_1 \pi \sqrt{t}), \quad && d = 1, \\ 1 - G_2(t) &\sim N/(-a(0)\gamma_2 \ln t), \quad && d = 2, \\ 1 - G_2(t) &\sim C_{N,d}, \quad && d \geqslant 3, \end{aligned}$$

where $C_{N,d} > 0$.

By virtue of this theorem the distribution time outside the sources are nonexponential, and therefore we have:

Corollary 1. *The random walk under consideration is Non-Markovian.*

3. Non-Markovian Branching Random Walks

Let us consider a finite set of branching sources located at the lattice points $x_1, x_2, \ldots, x_N \in \mathbf{Z}^d$, $d \ge 1$. We assume that the initial particle being outside the sources performs a continuous time random walk on $\mathbf{Z}^d$ until reaching one of the sources. The branching of particles at every point x_i, $i = 1, 2, \ldots, N$, is described by the branching processes with continuous time with the infinitesimal generation function of transition intensities $f(u) := \sum\limits_{n=0}^{\infty} b_n u^n$, where $b_n \geq 0$ for $n \neq 1$, $b_1 < 0$ and $\sum\limits_n b_n = 0$. We also assume that $\beta_r := f^{(r)}(1) < \infty$, $r \in \mathbf{N}$, and $\beta := \beta_1$. Let $\mu_t(y)$ be the number of particles at time t at the point $y \in \mathbf{Z}^d$. Then, the condition that at time $t = 0$ the system consists of a single particle at the point x amounts to $\mu_0(y) = \delta_x(y)$.

Now, let us consider BRWs on $\mathbf{Z}^d$ with State I and State II. In this case the evolution of a particle system in time may be described by the two-dimensional vector $\left(\sum_{i=1}^N \mu_t(x_i), \eta_t\right)$, where $\eta_t = \sum_{y \neq \{x_i\}_{i=1}^N} \mu_t(y)$, is the number of particles outside the branching sources. This construction is a more complicated example of the presented in Sec. 2 non-Markovian model based on random walks.

The present investigation proposes general methods to study non-Markovian models of supercritical BRW on $\mathbf{Z}^d$ for which an exponential growth of the particle number in every lattice point is observed. In the BRW models with finitely many sources of equal intensity, as was shown in [6], there arise multipoint perturbations of the symmetric random walk generator $\mathcal{A}$ which have the form

$$\mathcal{H}_\beta = \mathcal{A} + \beta \sum_{i=1}^{N} \Delta_{x_i},$$

where $x_i \in \mathbf{Z}^d$, $\mathcal{A} : l^p(\mathbf{Z}^d) \to l^p(\mathbf{Z}^d)$, $p \in [1, \infty]$, is the symmetric operator acting by formula

$$(\mathcal{A}u)(z) := \sum_{z' \in \mathbf{Z^d}} a(z - z')u(z'),$$

$\Delta_x = \delta_x \delta_x^T$, $\delta_x = \delta_x(\cdot)$ denotes the column vector on the lattice which is equal to 1 at the point x and to 0 at the other points. The perturbation $\beta \sum_{i=1}^N \Delta_{x_i}$ of the operator $\mathcal{A}$ can result in the appearance of positive eigenvalues of the operator $\mathcal{H}_\beta$. Let λ_0 be the leading positive eigenvalue of $\mathcal{H}_\beta$ then for $t \to \infty$ both the numbers of particles at the sources and their total number $\mu_t := \left(\sum_{i=1}^N \mu_t(x_i)\right) + \eta_t$ grow exponentially:

Theorem 2. *Let λ_0 be the leading positive eigenvalue of $\mathcal{H}_\beta$ then, in the sense of convergence of all moments,*

$$\begin{aligned} \lim_{t \to \infty} \left(\sum_{i=1}^{N} \mu_t(x_i) \right) e^{-\lambda_0 t} &= \psi(x_i, x_2, \cdots, x_N)\xi, \\ \lim_{t \to \infty} \mu_t e^{-\lambda_0 t} &= \xi, \end{aligned}$$

where $\psi(x_i, x_2, \cdots, x_N)$ is a function and ξ is a nondegenerate random variable.

Let us note that the limit relations in the last theorem can be meant also in the sense of convergence in distribution. The conditions under which the distribution ξ is uniquely determined by its moments were established using the Carleman criterium, see, e.g. [4]. Sufficient conditions for the exponential growth of the number of particles at the sources were obtained, e.g., in [6]. Thus, the methods developed in [6] are suitable for studying the offered non-Makrovian models based on supercritical BRWs with a finite number of branching sources.

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References

1. *Koroljuk V. S., Turbin A. F.* Matematicheskie osnovy fazovogo ukrupneniya slozhnykh sistem. – "Naukova Dumka", Kiev, 1978. (in Russian)
2. *Yarovaya E. B.* Spectral properties of evolutionary operators in branching random walk models // Math. Notes. – 2012. – Vol. 92, no. 1–2. – P. 115–131.
3. *Yarovaya E. B.* Non-markovian models of branching random walks // Proceedings of the III Symposium on semi-Markov models: Theory and Applications, Cagliary, June 17-19, 2009. – Cagliary, Italy, 2009. – P. 1–4.
4. *Yarovaya E. B.* Branching random walks in a heterogeneous environment. – Moscow : Center of Applied Investigations of the Faculty of Mechanics and Mathematics of the Moscow State University, 2007. (in Russian)
5. *Feller W.* An introduction to probability theory and its applications. – New York: John Wiley & Sons Inc., 1970, 1971. – Vol. I, II of Second edition.
6. *Yarovaya E. B.* Positive discrete spectrum of the evolutionary operator of supercritical branching walks with heavy tails // Methodology and Computing in Applied Probability. – 2016. – P. 1–17.