

# Skew Brownian motion with dry friction: The Pugachev–Sveshnikov equation approach

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**Abstract.** The Brownian motion with dry friction is one of the simplest but very common stochastic processes, also known as the Brownian motion with two valued drift, or the Caughey–Dienes process. This process appears in many applied fields, such as physics, mechanics, etc. as well as in mathematics itself. In this paper we are concerned with a more general process, skew Brownian motion with dry friction. We study the probability distribution of this process and its occupation time on the positive half line. The Pugachev–Sveshnikov equation approach is used.

**Keywords:** dry friction, skew Brownian motion, occupation time, local time, Pugachev–Sveshnikov equation, Riemann boundary value problem.

## 1. Introduction

Brownian motion plays an important role in statistical physics and other applied areas of science. The classical physical theory of Brownian motion was developed by Einstein and Smoluchowski [1] in the beginning of 20th century. Their model is based on the assumption that a Brownian particle is weightless, that mathematically leads to the Wiener process. The major drawback of this model is that the trajectories are continuous but nowhere differentiable, so velocity cannot be defined. A refined theory that gets over this issue by introducing particles' inertia was developed by Ornstein and Uhlenbeck [2] later on.

In both of the theories, a Brownian particle is driven by the random force resulting from collision of molecules, and the viscous friction takes place, that is, the force is proportional to the velocity of the particle. This is not always the case. A well-known example of that is the dry (Coloumb) friction of macroscopic materials. For this type of friction the resistive force is independent of the speed but depends on the direction of motion.

Behavior of the mechanical systems with dry friction under random excitation was first studied by Caughey and Dienes [3] in 60s. The Caughey–Dienes process is similar to the Ornstein–Uhlenbeck one up to the replacement of the viscous friction by dry, and its applications include control theory [4], seismic mechanics [5], communication systems theory [6], radio physics [7], and nonlinear stochastic dynamics [8]. It should be mentioned that this process also appears in purely mathematical papers [9–11]. In 2000s, there was another wave of interest in studying the Caughey–Dienes

process [12], that exist to this day. Some additional publications on the subject can be found in author's work [13].

The present paper deals with the so-called skew Caughey–Dienes process, or the skew Brownian motion with dry friction (SBM with dry friction), which generalizes Brownian motion with dry friction in the same way the skew Brownian motion generalizes the Wiener process [14].

## 2. Main section

For  $\eta \in (-1, 1)$  we define skew Brownian motion with dry friction  $X(t)$  as a unique strong solution [14] of the following equation

$$dX(t) = -2\mu \operatorname{sign}(X(t)) dt + \eta dL_X^0(t) + \sqrt{2}dW(t), \quad t > 0, \quad X(0) = 0.$$

By  $W(t)$  we denote a standard Wiener process starting at zero, and  $L_X^0(t)$  is the symmetric local time of the semimartingale  $X(t)$  at the level zero

$$L_X^0(t) = \lim_{\varepsilon \rightarrow +0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(-\varepsilon, +\varepsilon)}(X(s)) d[X]_s,$$

where  $[X]_s = 2s$  is the quadratic variation of  $X(s)$ . Further, we will be interested in the positive half-line occupation time of  $X(t)$

$$\mathcal{I}(t) = \int_0^t \mathbf{1}_{(0, +\infty)}(X(s)) ds.$$

One can think of  $X(t)$  and  $\mathcal{I}(t)$  as of the components of the vector diffusion process  $(X(t), \mathcal{I}(t))$  governed by the system of SDEs

$$\begin{cases} dX(t) = -2\mu \operatorname{sign}(X(t)) dt + \eta dL_X^0(t) + \sqrt{2}dW(t), \\ d\mathcal{I}(t) = \mathbf{1}_{(0, +\infty)}(X(s)) ds, \quad X(0) = \mathcal{I}(0) = 0. \end{cases} \quad (1)$$

In what follows, we derive the explicit formulas for the probability density function of (1), following ideas from [15]. Usually for this purpose one uses the Fokker–Planck–Kolmogorov equation or random walks approximation. We use an alternative approach based on the characteristic function method that manifests in use of the Pugachev–Sveshnikov singular integral differential equation.

It can be shown [15] that the characteristic function  $E(z_1, z_2; t)$  of the process  $(X(t), \mathcal{I}(t))$  satisfies the equation

$$\frac{\partial E}{\partial t} + (z_1^2 - iz_2/2)E + (2\mu z_1 - z_2/2)\hat{E} - 2i\eta z_1\Psi_0 = 0, \quad E(z_1, z_2; 0) = 1, \quad (2)$$

where we adopt the short notation  $\hat{E}(z_1, z_2; t)$  and  $\Psi_0(z_2, t)$ :

$$\hat{E} = \frac{1}{\pi} \text{v.p.} \int_{-\infty}^{+\infty} \frac{E|_{z_1=s}}{s - z_1} ds, \quad \Psi_0 = \frac{1}{2\pi} \text{v.p.} \int_{-\infty}^{+\infty} E|_{z_1=s} ds. \quad (3)$$

For  $\text{Im } \zeta \neq 0$  let us introduce the Cauchy-type integral  $\Phi(\zeta, z_2; t)$  and its limit values  $\Phi^\pm(z, z_2; t)$  on the real axis from upper and lower half-planes (with respect to the first argument):

$$\Phi(\zeta, z_2; t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{E|_{z_1=s}}{s - \zeta} ds, \quad \Phi^\pm(z, z_2; t) = \lim_{\zeta \rightarrow z \pm i0} \Phi(\zeta, z_2; t), \quad \text{Im } z = 0.$$

It is well known that  $\Phi(\cdot, z_2; t)$  is analytic when  $\text{Im } \zeta \neq 0$ , and that  $\Phi^\pm$  satisfy Sokhotski–Plemelj formulas when  $\text{Im } z = 0$ :

$$\Phi^+ - \Phi^- = E, \quad \Phi^+ + \Phi^- = -i\hat{E}. \quad (4)$$

Clearly, one can rewrite (2) in terms of  $\Phi^\pm$ , that gives a Riemann boundary value problem. Applying then Laplace transform with respect to  $t$ , and denoting its argument by  $p$ , we get to the formula

$$(z_1^2 + 2\mu i z_1 + p - i z_2) \tilde{\Phi}^+ - i \eta z_1 \tilde{\Psi}_0 - \frac{1}{2} = (z_1^2 - 2\mu i z_1 + p) \tilde{\Phi}^- + i \eta z_1 \tilde{\Psi}_0 + \frac{1}{2}. \quad (5)$$

The Laplace transforms are labeled with the tildes above the functions.

Note that the left-hand side of (5) can be analytically continued for all  $z_1 \in \mathbb{C}$  such that  $\text{Im } z_1 > 0$ , also the right-hand side can be analytically continued for all  $z_1 \in \mathbb{C}$  such that  $\text{Im } z_1 < 0$ . Since they match when  $\text{Im } z_1 = 0$ , they turn out to be elements of the same entire function of argument  $z_1 \in \mathbb{C}$ . Assuming that  $\Phi^\pm(z, z_2; t) = O(\frac{1}{|z|})$  when  $z \rightarrow \infty$  for  $\text{Im } z \geq 0$ , by generalized Liouville's theorem one can realize that this entire function is actually linear:  $G_0(z_2, t) + z_1 G_1(z_2, t)$ . This leads to the equality

$$\tilde{\Phi}^\pm = \frac{G_0 + z_1 G_1 \pm i \eta z_1 \tilde{\Psi}_0 \pm 1/2}{z_1^2 \pm 2i\mu z_1 + p - (1 \pm 1)iz_2/2}. \quad (6)$$

Note that the denominator in (6) has zeros  $i\nu^\pm = i(-\mu \pm \sqrt{\mu^2 + p - iz_2})$  and  $i\kappa^\pm = i(\mu \pm \sqrt{\mu^2 + p})$  such that  $\text{Im}(i\nu^\pm) \geq 0$  and  $\text{Im}(i\kappa^\pm) \geq 0$ . At the same time,  $\tilde{\Phi}^\pm$  should be analytic in upper and lower half-planes, therefore, the singularities at  $i\nu^+$  and  $i\kappa^-$  should be removable. This gives a system of linear equations to determine  $G_0$  and  $G_1$ . Also, taking into account the definition of  $\Psi_0$  in (3) and performing integration in (6) one

can get that  $\tilde{\Psi}_0 = -iG_1$ . After that, the system of linear equations for  $G_0$  and  $G_1$  can be written in the following form

$$G_0 + i\nu^+(1 + \eta)G_1 + 1/2 = 0, \quad G_0 + i\kappa^-(1 - \eta)G_1 - 1/2 = 0. \quad (7)$$

Finally, substituting  $G_0$  and  $G_1$  from (7) into (6), one can get the final expression for  $\tilde{E}$  using the first of the Sokhotski–Plemelj formulas (4). Particularly, after necessary simplifications one can get the Laplace transform of the characteristic function of  $X(t)$  and  $\mathcal{I}(t)$ :

$$\begin{aligned} \tilde{E}_X(z, p) &= \tilde{E}(z, 0; p) = \frac{1}{2i\kappa^-} \left( \frac{1 + \eta}{z + i\kappa^+} - \frac{1 - \eta}{z - i\kappa^-} \right), \\ \tilde{E}_{\mathcal{I}}(z, p) &= \tilde{E}(0, z; p) = \frac{(1 + \eta)\kappa^+ - (1 - \eta)\nu^-}{\nu^- \kappa^+ ((1 - \eta)\kappa^- - (1 + \eta)\nu^+)}. \end{aligned}$$

Cumbersome but straightforward computations will immediately lead us to the final expressions for the corresponding probability density functions (PDFs).

**Theorem.** *The PDF of  $X(t)$ , the steady-state PDF of  $X(t)$ , and the PDF of the positive half-line occupation time  $\mathcal{I}(t)$  have the following form:*

$$\begin{aligned} f_X(x, t) &= \left( \frac{1}{\sqrt{\pi t}} e^{-\frac{(|x|+2\mu t)^2}{4t}} + \mu e^{-2\mu|x|} \text{Erfc} \left| \frac{|x| - 2\mu t}{2\sqrt{t}} \right| \right) \cdot \begin{cases} \alpha, & x > 0, \\ 1 - \alpha, & x < 0, \end{cases} \\ f_X^\infty(x) &= f_X(x, +\infty) = 2\mu e^{-2\mu|x|} (\alpha \mathbf{1}_{(0, +\infty)}(x) + (1 - \alpha) \mathbf{1}_{(-\infty, 0]}(x)), \\ f_{\mathcal{I}}(y, t) &= \frac{4e^{-\mu^2 t}}{\pi \sqrt{y(t-y)}} \int_0^{+\infty} \int_0^{+\infty} \chi(2\sqrt{y}s_1, 2\sqrt{t-y}s_2) s_1 s_2 e^{-s_1^2 - s_2^2} ds_1 ds_2, \end{aligned}$$

where  $0 < y < t$ , the function  $\text{Erfc}(\cdot)$  is the complementary error function,  $\alpha = (1 + \eta)/2$ , and

$$\begin{aligned} \chi(s_1, s_2) &= \frac{1 - \eta}{1 + \eta} e^{-\mu(s_1 + \frac{\eta-3}{\eta+1}s_2)} \chi^+(s_1, s_2) + \frac{1 + \eta}{1 - \eta} e^{-\mu(\frac{\eta+3}{\eta-1}s_1 + s_2)} \chi^-(s_1, s_2), \\ \chi^+(s_1, s_2) &= \mathbf{1}_{(0, +\infty)}((1 + \eta)s_1 - (1 - \eta)s_2), \quad \chi^-(s_1, s_2) = 1 - \chi^+(s_1, s_2). \end{aligned}$$

### 3. Conclusions

We derived explicit formulas for the probability density function of the Brownian motion with dry friction and its occupation time on the positive half-line, that generalizes known results for the regular Caughey–Dienes process. In fact, more general result was obtained for the Laplace transform of the joined characteristic function. Essentially, our approach is based on the reduction to a Riemann boundary value problem, and clearly it can be used to find the characteristics of more general SDEs with piecewise linear coefficients and local time.

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