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# Joint distributions of synchronization models

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**Abstract.** We consider Markov models of multicomponent systems with synchronizing interaction. Under natural regularity assumptions about the message routing graph, they have nice longtime behavior. We are interested in limit probability laws related to the steady state viewed from the center-of-mass coordinate system.

**Keywords:** stochastic synchronization models, long-time behavior, Markov processes.

## 1. Introduction

The study of stochastic synchronization models is motivated by many applications in computer science [1, 2] and other domains. One of the most interesting problems is the synchronization of local clocks in asynchronous networks [3, 4]. A common feature of synchronization models is the use of time-stamped messages. Dynamics of these systems is a superposition of independent random evolutions of components and an event driven interaction resulting from an information exchange between components. Such models are very similar to traditional queueing networks but the synchronizing jump interaction between components also lets to consider them as a special class of interaction particle systems. There are many results on longtime behaviour of symmetric synchronization models [5, 6].

A general nonsymmetric Markovian synchronization model was introduced and studied in [7, 8]. Consider a network of $N$ nodes and denote by $x_j \in \mathbb{R}^d$ the state of the node $j$. The dynamics of the network is a stochastic process $x(t) = (x_1(t), \ldots, x_N(t)) \in \left(\mathbb{R}^d\right)^N$, $t \in \mathbb{R}_+$. The evolution of $x(t)$ is composed of two parts called respectively a free dynamics and synchronizing jumps. The free dynamics means that between successive epochs of interaction the components $x_j(t)$ evolve independently and their increments follow increments of some $\mathbb{R}^d$-valued processes $x_j^\circ(t)$. It is assumed that nodes share information about each other by sending and receiving messages. A message flow from node $j_1$ to node $j_2$ is Poissonian with rate $\alpha_{j_1 j_2}$. Messages reach their destinations instantly. A message sent at time $T$ from $j_1$ to $j_2$ forces the destination node $j_2$ to adjusts its state to the value $x_{j_1}$: $x_{j_2}(T + 0) = x_{j_1}(T)$. These adjustments are interpreted as synchronizing jumps. The initial configuration $x(0)$ is chosen independently from other sources of randomness. We also assume that the communication graph is strongly connected, i.e., that any pair of nodes can be connected by a directed path composed of arcs $(j, k)$ such that $\alpha_{jk} > 0$ (the *connectivity assumption*).

In [7–9] as well as in the present note we will focus on Lévy-driven synchronization models, i.e., the free dynamics of the $j$th component is assumed to be a Lévy process $x_j^\circ(t) \in \mathbb{R}^d$ with characteristic exponent $-\eta_j$. So $x(t)$ is a continuous time Markov process. We refer to [9] for the explicit form of its generator.

## 2. Known results

Here is a short list of selected results proved in [8, 9]. In the center-of-mass coordinate system

$$y_j(t) = x_j(t) - M(t), \qquad M(t) := N^{-1} \sum_{j=1}^{N} x_j(t)\,,$$

there exists a limit in distribution of $y(t) = (y_1(t), \ldots, y_N(t))$ as $t \to \infty$. In this sense we observe a stochastic synchronization phenomenon in the long-time behaviour of the model. For non-trivial free dynamics there exists no limit in distribution of $x(t)$ as $t \to \infty$.

If all LPs $x_j^\circ(t)$ are stable with index $\beta$ then the center-of-mass $M(t)$ has a simple asymptotic behavior. Rescaling it as $m(t) := M(t)/t^{1/\beta}$ and applying Theorem 2 from [8] we obtain that $m(t)$ converges in distribution, $m(t) \xrightarrow{d} m(\infty)$, moreover, $\psi_{y(t),m(t)}(u, \rho)$, the joint characteristic function (CF) of $y(t)$ and $m(t)$, has the following limit as $t \to \infty$

$$\forall u \in \left(\mathbb{R}^d\right)^N, \rho \in \mathbb{R}^d \qquad \psi_{y(t),m(t)}(u, \rho) \to \psi_{y(\infty)}(u)\psi_{m(\infty)}(\rho).$$

Hence the vectors $y(t)$ and $m(t)$ become asymptotically independent.

Below we are intended to find out properties of the steady state $y(\infty)$ for Lévy-driven synchronization models under additional assumptions on network topology. Some results on limit distributions of $r_{jk}(t) = x_k(t) - x_j(t)$ were presented in [6, 8, 9]. It was shown that the class of limit distributions is rich enough to include, in particular, multivariate asymmetric Laplace distributions [10], Linnik [11] and bilateral matrix-exponential distributions [12].

Now consider a stochastic process $\vec{r}(t) = \left(r_{jk}(t),\ j, k = \overline{1, N}\,,\ j \neq k\right)$ with values in $\left(\mathbb{R}^d\right)^{(N-1)N}$. Similarly to Theorem 2 in [9] we get that $\vec{r}(t)$ has a limit in law: $\vec{r}(t) \xrightarrow{d} \vec{r}(\infty)$. The aim of the present short note is to obtain new explicit formulae for marginals of $\vec{r}(\infty)$.

For $\mathbb{R}^d$-vectors $\nu = \left(\nu^{(1)}, \ldots, \nu^{(d)}\right)$ and $\rho = \left(\rho^{(1)}, \ldots, \rho^{(d)}\right)$ we denote by $\nu \cdot \rho = \sum_l \nu^{(l)} \rho^{(l)}$ their scalar product. Let i be the imaginary unit, $\mathrm{i}^2 = -1$. The limit characteristic functions (CFs)

$$\lim_{t\to\infty} \psi_{r_{jk}(t)}(\nu) = \lim_{t\to\infty} \mathsf{E} \exp \mathrm{i}\nu \cdot r_{jk}(t)$$

will be denoted by $\varphi_{(jk)}(\nu)$, $\nu \in \mathbb{R}^d$. We are going to study new synchronization models with special communication graphs.

## 3. The $R_N$-model

Consider a network with $N$ identical nodes and the rotationaly invariant topology, i.e., $\eta_j = \eta$ and $\alpha_{jk} = \alpha_{k-j}$ $(j \neq k)$ where subtraction is taken modulus $N$. Recall that $-\eta$ is the characteristic exponent of some Lévy process hence $\mathrm{Re}\,\eta(\nu) \geq 0$ for all $\nu \in \mathbb{R}^d$. Put $a_0 = 0$ and $a_m = \alpha_{-m} + \alpha_m$ for $m \neq 0$. Define $\widehat{a}_n := \sum_{m=0}^{N-1} a_m s_n^m$, the discrete Fourier transform of the vector $(a_m)$, where $s_n = \exp(2\pi \mathrm{i} n/N)$.

**Theorem 1** *The CFs $\varphi_{(jk)}(\nu)$ have the form $\varphi_{(jk)}(\nu) = H_{k-j}^{(N)}(2\,\mathrm{Re}\,\eta(\nu))$ with functions $H_m^{(N)}(b) = F_m^{(N)}(b)/F_0^{(N)}(b)$, $m = 1, \ldots, N-1$, where*

$$F_m^{(N)}(b) = \sum_{n=0}^{N-1} \frac{s_m^{-n}}{b + \widehat{a}_0 - \widehat{a}_n}, \qquad F_0^{(N)}(b) = \sum_{n_1=0}^{N-1} \frac{1}{b + \widehat{a}_0 - \widehat{a}_{n_1}}.$$

*The functions $H_m^{(N)}(b)$ are the Laplace-Stieltjes transforms (LSTs) of some probability distributions $\mu_{N,m}$ supported on $\mathbb{R}_+$.*

In other words, the theorem states that the laws of $r_{jk}(\infty)$ belong to so-called compound distributions and gives a convenient tool for characterizing the limit CFs $\varphi_{(jk)}(\nu)$.

Sometimes we drop the explicit $N$-dependence notation of $\varphi_{(jk)}$ and $H_m$ when there is no confusion. It is readily seen that $F_m^{(N)}(b)$, $F_0^{(N)}(b)$ and $H_m(b)$ are rational functions of $b \in \mathbb{C}$ and $\overline{H_k(b)} = H_k(\bar{b})$. Evidently, there are no poles of $H_m(b)$ at points $\widehat{a}_n - \widehat{a}_0$. The degree of any of the rational functions $H_m(b)$ does not exceed $N-1$. In fact, by using the symmetry property $a_{N-j} = a_j$ it is easy to show that $H_m(b) = H_{N-m}(b)$ and that $\deg H_m \leq [N/2]$. If the number of *distinct* values among the Fourier coefficients $\widehat{a}_0, \widehat{a}_1, \ldots, \widehat{a}_{[N/2]}$ is *less* than $[N/2]+1$ then $\deg H_m < [N/2]$. We see also that $\varphi_{(jk)}(\nu)$ are real and so $r_{jk}(\infty)$ have symmetric laws.

The proof of this theorem is longer than the allowed size of this note. The rest of the paper is devoted to examples where the theorem leads to interesting explicit results.

## 4. Example: the $R_2$ and $R_3$-models

The 2-node network $R_2$ depends on a single parameter $\alpha_1 > 0$. We have $s_0 = 1$, $s_1 = -1$, $a_1 = 2\alpha_1$, $\widehat{a}_0 = -\widehat{a}_1 = 2\alpha_1$

$$F_1^{(2)}(b) = \frac{1}{b} - \frac{1}{b + 4\alpha_1}\,, \quad F_0^{(2)}(b) = \frac{1}{b} + \frac{1}{b + 4\alpha_1}\,, \quad H_1^{(2)}(b) = \frac{2\alpha_1}{b + 2\alpha_1}\,.$$

Thus $H_1^{(2)}$ is the LST of the exponential law. For the $R_3$-model there are two parameters $\alpha_1$ and $\alpha_2$ and $\alpha_1 + \alpha_2 > 0$ is a non-degeneracy condition. For $N = 3$ we have $a_1 = a_2 = \alpha_1 + \alpha_2$ and the answer is essentially the same: $H_1^{(3)}(b) = H_2^{(3)}(b) = (\alpha_1 + \alpha_2)/(b + \alpha_1 + \alpha_2)\,$.

Taking, for example, $\eta(\nu) = \frac{1}{2}\sigma^2\,|\nu|^2 - \mathrm{i}\,v\cdot\nu$, $c\,|\nu|$ or $(c\,|\nu|)^\beta$, where $\beta \in (0,2)$, $c > 0$, (i.e., the common free dynamics of nodes $x^\circ(t)$ is chosen as the $d$-dimensional Brownian motion with constant drift $v \in \mathbb{R}^d$, the Cauchy process or the $\beta$-stable Lévy process) we obtain as distributions of $r_{jk}(\infty)$ the multidimensional asymmetric Laplace probability law [10] or distributions from the Linnik class.

## 5. Example: the $R_4$-model

Now $a_1 = a_3 = \alpha_1 + \alpha_3$, $\quad a_2 = 2\alpha_2$. There are two *simple cases.*

**1**) If $a_k = a$ for all $k = 1, 2, 3$ then $H_m^{(4)}(b) = \frac{a}{b+a}\,$.

**2**) The assumption $\alpha_1 = \alpha_3 = 0$ is equivalent to $a_1 = a_3 = 0$. The connectivity assumption is violated. The network is splitted into two parts $\{1,3\}$ and $\{2,4\}$ which are evolving independently.

In the *general case* straightforward calculations show that there exist $q_2 > q_1 > 0$ such that for all $k = 1, 2, 3$

$$H_k^{(4)}(b) = g_1^{(4,k)}\frac{q_1}{b + q_1} + g_2^{(4,k)}\frac{q_2}{b + q_2}$$

where reals $g_1^{(4,k)}$, $g_2^{(4,k)}$ are not necessarily positive and satisfy $g_1^{(4,k)} + g_2^{(4,k)} = 1$. Thus $H_k^{(4)}(b)$ are linear combinations of LSTs of two exponential distributions.

For the general $R_5$-model the distributions $\mu_{5,m}$ are again mixtures of two exponential laws. Networks $R_N$ with $N \geq 6$ provide much more intriguing structure of the probability laws $\mu_{N,m}$. We will devote to them a separate publication.

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