UDC 519.2

# Limit theorems for Additive Functionals of Semi-Markov Processes

**A. Oprisan***

* *Department of Mathematics and Statistics,*
*Canisius College,*
*Buffalo NY, USA*

**Abstract.** We consider a class of additive functionals of ergodic semi-Markov processes and their associated Markov renewal processes that have a martingale decomposition representation. We prove an invariance principle and show that the corresponding empirical processes converge almost surely to the Wiener process (almost sure functional central limit theorem).

**Keywords:** additive functionals, almost sure central limit theorem, invariance principle, Markov process, Markov renewal process, semi-Markov process.

## 1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ the standard right-continuous filtration, $(E, \mathcal{E})$ a complete separable metric space and $Q(x, A \times \Gamma)$, $x \in E$, $A \in \mathcal{E}$, $\Gamma \in \mathcal{B}_+$ a semi-Markov kernel on $(E \times \mathbb{R}_+, \mathcal{E} \times \mathcal{B}_+)$ ($\mathcal{B}_+$ is the Borel $\sigma$-algebra of $\mathbb{R}_+$). Define a time-homogeneous semi-Markov process $\{X(t) : t \geq 0\}$ with semi-Markov kernel $Q$ and with values in $(E, \mathcal{E})$ as follows. Given a jump-type Markov process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in $(E, \mathcal{E})$ and $0 = \tau_0 < \tau_1 < ....$ the jump times, then one defines a discrete-time Markov chain $\{X_n, n = 0, 1, ...\}$ by $X_n = X(\tau_n)$ having transition probability kernel $P(x, dy) := Q(x, dy \times [0, \infty))$. Given a probability measure $\mu$ on $(E, \mathcal{E})$, define the probability measures $\mathbb{P}_\mu$ by

$$\mathbb{P}_\mu(A) = \mu\mathbb{P}(A) = \int_E \mu(dx)\, p(x, A), \quad x \in E,\, A \in \mathcal{F},$$

and the transition probability operator $P$,

$$P\varphi(x) := \mathbb{E}[\varphi(X_{n+1}) \,|\, X_n = x] = \int_E p(x, dy)\varphi(y).$$

Let $P^n$ be the $n$-step transition operator corresponding to the $n$-step transition probability $p^n(x, A)$. The stochastic process $\{(X_n, \tau_n), n \geq 0\}$ is called the embedded Markov renewal process with renewal times $\tau_n$ and for any $n \geq 0$, $A \in \mathcal{E}$, and $\Gamma \in \mathcal{B}_+$,

$$\mathbb{P}(X_{n+1} \in A,\, \tau_{n+1} - \tau_n \in \Gamma \,|\, X_n = x) = Q(x, A \times \Gamma)$$

Let $N(t) = \max\{n : \tau_n \le t\}$ be the point process that counts the jumps of $X$ in the time interval $(0, t]$. The semi-Markov process $\{X_t : t \ge 0\}$ is defined by setting $X_t = X_{N(t)}$. Denote $\theta_n = \tau_n - \tau_{n-1}$, $n \ge 1$ the inter-jumps times. The random variable $\theta_n$ is also called the sojourn time in the state $X_n$ and given $\{X_n, n \ge 0\}$ the random variables $\{\theta_n, n \ge 0\}$ are mutually independent. Denote $F_x(t) = \mathbb{P}(\theta_{n+1} \le t \,|\, X_n = x) = Q(x, E \times [0, t])$ the sojourn distribution in the state $x \in E$ and $\lambda(x, t)$ its hazard rate function

$$F_x(t) = 1 - \exp\{-\int_0^t \lambda(x, u)\, du\}.$$

Let's define the mean sojourn time by

$$\tilde{m} := \int_E \nu(dx) m(x) < \infty, \text{ where } m(x) := \int_0^\infty \bar{F}_x(t) dt,\ \bar{F}_x(t) = 1 - F_x(t).$$

The two-component process $\{(X_n, \theta_{n+1}), n \ge 0\}$ taking values in $E \times [0, \infty)$ is a Markov process, also called Markov renewal process and its transition probabilities are given in terms of the semi-Markov kernel

$$Q(x, A \times \Gamma) = \mathbb{P}(X_{n+1} \in A, \theta_{n+2} \in \Gamma \,|\, X_n = x).$$

In this paper we assume that the embedded Markov chain of the semi-Markov process satisfies the following assumptions:

A1. The semi-Markov process $X$ is regular, i.e. $(\forall)\, x \in E$, $(\forall)\, t \ge 0$, $\mathbb{P}_x(N(t) < \infty) = 1$;

A2. The Markov chain $\{X_n, n \ge 0\}$ is Harris ergodic with stationary distribution $\nu$, i.e.

$$\nu(A) = \int_E \nu(dx) p(x, A);$$

A3. The mean sojourn time in a state $x \in E$ is uniformly bounded;

A4. The family of sojourn times $\{\theta_x, x \in E\}$ is uniformly integrable, i.e.

$$\sup_{x \in E} \int_N^\infty \bar{F}_x(t)\, dt \to 0, \quad N \to \infty, \quad \text{and} \quad \sup_{x \in E} \int_N^\infty t\bar{F}_x(t)\, dt \to 0,\ N \to \infty.$$

## 2. Main section

Asymptotic results for additive functionals of semi-Markov processes including functional central limit theorems have been studied by many authors. For the discrete space state case we refer to [2] for a functional central limit theorem. In a general state space context we mention [4], where a functional central limit theorem for additive functionals of semi-Markov processes on a general state space is considered but the reward function $f$ is bounded. Note that for additive functionals of Markov processes, an important result is due to R.N. Bhattacharya [1] and in this paper we want to generalize it to semi-Markovian case.

Let $\{X_t,\, t \geq 0\}$ be a semi-Markov process with stationary probability measure $\pi$ and $f : E \to \mathbb{R}_+$ be a Borel function. Define the additive functional by

$$W_t := \int_0^t f(X_s)\, ds$$

Then

$$W_t = \sum_{k=1}^{N(t)} f(X_{k-1})\theta_k + (t - \tau_{N(t)}) f(X_{N(t)}).$$

Let $\mathcal{D}([0,\infty), E)$ be the space of càdlàg functions $\{f : [0,\infty) \to E\}$ and let $\{\mathbf{W}(t),\, t \geq 0\}$ be standard Wiener process on $\mathcal{D}([0,\infty)$.

**Lemma 2.1** *Assume that $\{X_t,\, t \geq 0\}$ is an ergodic semi-Markov process with ergodic distribution $\pi$ and $\{X_n,\, n \geq 0\}$ is its embedding Markov chain with stationary distribution $\nu$. Let $f \in L^2(\pi)$ be such that*

*(i)* $\int_E f d\pi = 0$,

*(ii) there exists $0 < C < \infty$ such that $d\mu P^k \leq C\, d\nu$ for any $k \in \mathbb{N}$ and $\int_{\{x: f^2(x) > n\}} f^2(x)\nu(dx) \leq \varphi(n)$ where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\lim_{x\to\infty} \varphi(x) = 0$.*

*Let $W_t^n = \frac{1}{\sigma\sqrt{n}} \int_0^{nt} f(X_u)\, du$ and $\tilde{W}_t^n = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{N(nt)} f(X_{k-1})\theta_k$. Then the random processes $W_t^n$ and $\tilde{W}_t^n$ have the same limiting distribution.*

*Proof:* It is enough to prove that $\sup_{0 \leq t < \infty} |W_t^n - \tilde{W}_t^n|$ converges in probability to zero. This follows due to integrability and regularity conditions. ■

An important step in proving the functional central limit theorem and the almost sure central limit theorem is a martingale decomposition for the Markov renewal process.

**Theorem 2.2** *Let $\{X_t,\, t \geq 0\}$ be an ergodic semi-Markov process with initial distribution $\mu$ and unique invariant measure $\pi$ and $\{X_n,\, n \geq 0\}$ its embedded Markov chain with invariant probability measure $\nu$ . Consider $f \in \mathbb{L}^2(\pi)$ satisfying the following conditions:*

*(i)* $\int_E f(x)\pi(dx) = 0$

*(ii)* $\|P^k f\|_{L^2(\nu)} \leq \rho^k \|f\|_{L^2(\nu)}$ *for some* $0 < \rho < 1$, $k \in \mathbb{N}$

*(iii)* *there exists* $0 < C < \infty$ *such that* $d\mu P^k \leq C\, d\nu$ *for any* $k \in \mathbb{N}$ *and* $\int_{\{x: f^2(x) > n\}} f^2(x)\nu(dx) \leq \exp(-\varphi(n))$ *for* $n$ *large, with* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is such that* $\lim_{x\to\infty} \frac{\varphi(x)}{\log x} = \infty$.

*(iv)* $|P^k f(x)| \leq Cn$, *whenever* $|f(x)| \leq n$ *for some* $1 < C < \infty$ *and* $n$ *sufficiently large.*

*Then the additive functional of the Markov renewal process satisfies the following martingale decomposition:*

$$S_n(f) = \sum_{k=1}^{n} f(X_{k-1})\theta_k = M_n + R_n$$

*where* $M_n$ *is a local* $L^2$*-martingale with respect to the filtration* $\mathcal{F}_n = \sigma\{X_k, 0 \leq k \leq n\}$ *and the remainder term goes in probability to zero and satisfies*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\left\{ \frac{\sup_{1\leq k\leq n} R_k^2}{n} > \varepsilon \right\} = -\infty. \tag{1}$$

*Proof:* Since $\{(X_{n-1}, \theta_n), n \geq 1\}$ is the corresponding renewal Markov process associated to the semi-Markov process, it is stationary with probability invariant measure $\tilde{\nu} = \nu \cdot F$, $\tilde{\nu}(dy \times ds) = \nu(dy)F_y(ds)$.

Define the measurable function $g : E \times \mathbb{R}_+ \to \mathbb{R}_+$ as $g(y, s) = f(y)s$, and let $S_n(g) = \sum_{k=1}^{n} g(X_{k-1}, \theta_k)$. The martingale decomposition follows using the technique developed in [3], Theorem 3.1.

■

**Theorem 2.3** *Let* $\{X_t, t \geq 0\}$ *be a stationary ergodic semi-Markov process with invariant distribution* $\pi$ *and* $f \in L^2(E, \pi)$ *satisfying the assumptions of Theorem 2.2. Then the process* $W_t^n := \frac{1}{\sigma\sqrt{nm}} \int_0^{nt} f(X_s)\, ds$ *converges weakly to the standard Wiener process* $W$ *on* $\mathcal{D}([0,\infty), E)$.

*Proof:* According to Theorem 2.2, $S_t^n = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[nt]} f(X_{k-1})\theta_k = \frac{1}{\sigma\sqrt{n}} M_t^n + \frac{1}{\sigma\sqrt{n}} R_t^n$, with the remaining term converging almost surely to zero. Indeed, let $A_n = \{\omega : \sup_{t\in[0,\infty)} \frac{|R_t^n(\omega)|}{\sigma\sqrt{n}} \geq \varepsilon\}$. From (1) we get that $\mathbb{P}(A_n) \leq n^{-a_n}$ where $a_n$ is a sequence converging to infinity. Since $\sup_{n=1}^{\infty} n^{-a_n} < \infty$, Borel-Cantelli lemma implies that $A_n$ converges to zero $\mathbb{P}$-a.s.. The first term is a local $L^2$- martingale, and following a standard localizing technique the convergence in distribution to the Wiener measure on $\mathcal{D}([0,\infty), E)$ is reduced to the convergence in distribution for square integrable martingales. Since $\sup_{0\leq t<\infty} \left| \frac{N(nt)}{n} - \frac{t}{m} \right| \to 0$ as $n \to \infty$, with $m = \mathbb{E}_\pi(X_1)$, using Anscombe-Donsker Invariance principle for Markov chains, we get the limiting distribution of $\tilde{W}_t^n = \frac{1}{\sigma\sqrt{nm}} \sum_{k=1}^{N(nt)} f(X_{k-1})\theta_k$ to the Wiener process $W$. Based on Lemma 2.1, the conclusion follows. ■

**Theorem 2.4** *(almost sure central limit theorem) Let* $\{X_t,\, t \geq 0\}$ *be a semi-Markov process with an unique stationary distribution* $\pi$. *Consider the additive functional* $W_t^{(n)} = \frac{1}{\sigma\sqrt{nm}} \int_0^{nt} f(X_s)\, ds$ *and the corresponding empirical measure* $\mathbf{W_n}(\omega) = \frac{\mathbf{1}}{\log \mathbf{n}} \sum_{\mathbf{k=1}}^{\mathbf{n}} \frac{\mathbf{1}}{\mathbf{k}} \delta_{\mathbf{W^{(k)}}(\omega)}$. *Then, under the assumptions of Theorem 2.2,* $lim_{n\to\infty} \mathbf{W_n} = \mathbf{W}, \quad \mathbb{P}-\mathbf{a.e.}$, *where* $\mathbf{W}$ *is the Wiener measure on* $\mathcal{D}[0,\infty)$.

## 3. Conclusions

The functional central limit theorem and almost sure central limit theorem for additive functionals of Markov processes have been studied in our previous work [3], [5]. We generalized these results in the case of semi-Markov processes. Our approach is different than the methodologies used in other papers existing in the literature that studied the invariance principle for additive functionals of semi-Markov processes, and relies on the martingale decomposition of a renewal process associated with the semi-Markov process. This martingale decomposition is crucial in proving further that the functional central limit theorem admits an almost sure version.

## Acknowledgments

The work is partially supported by the Dean of Arts and Sciences Summer Grant, Canisius College.

## References

1. *Bhattacharya R.N.* On the functional central limit theorem and the law of the iterated logarithm for Markov processes // Z. Wahrsch. verw. Gebiete. — 1982. — Vol. 60. — P. 185–201.
2. *Glynn P.W., Haas P.J.* On functional central limit theorems for semi-Markov and related processes // Communications in Statistics - Theory and Methods. — 2004. — Vol. 33. — P. 487–506.
3. *Korzeniowski A., Oprisan A.* Large deviations for additive functionals of Markov processes // International Journal of Pure and Applied Mathematics. — 2009. — Vol. 53. — P.441–459.
4. *Limnios N.* Estimation of the stationary distribution of semi-Markov processes with borel state space // Statistics & Probability Letters. — 2006. — Vol. 76. — P. 1536–1542.

5. *Oprisan A., Korzeniowski A.* Large deviations via almost sure clt for functionals of Markov processes // Stochastic Analysis and Applications. — 2012. — Vol. 30. — P. 933–947.