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Numerical analysis of phase transitions in supercritical branching random walks

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Abstract. We consider a continuous-time symmetric supercritical branching random walk on a multidimensional lattice with a finite set of particle generation centers, i.e. branching sources. We construct the model with three branching sources located at the vertices of the simplex with positive or negative intensities and present branching random walk, where arbitrary number of the branching sources with positive or negative intensities are located in the vertices of the simplex. It is established that the amount of positive eigenvalues of the evolutionary operator, counting their multiplicity, does not exceed the amount of the branching sources with positive intensity, while the maximal of eigenvalues is simple.

Keywords: branching random walks, evolutionary operator, discrete spectrum, analytical methods in probability theory.

1. Introduction

The continuous-time branching random walk (BRW) with a finite number of branching sources situated at the points of the multidimensional lattice $\mathbb{Z}^d$ $(d \geq 1)$ is considered. The behaviour of the mean number of particles both at an arbitrary point and on the entire lattice can be described in terms of the evolutionary operator of a special type (e.g. [**?**]), which is a perturbation of the generator $\mathcal{A}$ of a symmetric random walk. This operator has form

$$\mathcal{H}_\beta = \mathcal{A} + \sum_{i=1}^{N} \beta_i \delta_{x_i} \delta_{x_i}^T, \quad x_i \in \mathbb{Z}^d,$$

where $\mathcal{A} : l^p(\mathbb{Z}^d) \to l^p(\mathbb{Z}^d),\ p \in [1, \infty]$ is a symmetric operator and $\delta_x = \delta_x(\cdot)$ denotes a column vector on the lattice taking the value one at the point x and zero otherwise. Branching occurs at some sources x_i and is defined by an infinitesimal generating functions $f_i(u) = \sum_{n=0}^{\infty} b_{i,n} u^n$ such that $\beta_{i,r} = f_i^{(r)}(1) < \infty$ for all $r \in \mathbb{N}$. The quantity $\beta_i \equiv \beta_{i,1}$ characterizing the intensity of x_i source.

General analysis of this operator was first done in [**?**]. The perturbation of the form $\sum_{i=1}^{N} \beta_i \delta_{x_i} \delta_{x_i}^T$ of the operator $\mathcal{A}$ may result in the emergence of positive eigenvalues of the operator $\mathcal{H}_\beta$ and the multiplicity of each of them does not exceed N. In [?] it was proved that for the case of equal β_i and finite variance of jumps the total multiplicity of all eigenvalues does not exceed N and the multiplicity of each eigenvalue of the operator $\mathcal{H}_\beta$ does not exceed $N-1$. In [?] the same was proved for the case of infinite variance of jumps and was demonstrated that the appearance of multiple lower eigenvalues in the spectrum of the evolutionary operator can be caused by a simplex configuration of branching sources.

2. Main section

We consider branching random walk with $p+n$ sources, that are located in the vertices of a simplex on $\mathbb{Z}^d$. P sources $x_1 ... x_p$ have intensity $\beta > 0$ and n sources $x_{p+1} ... x_{p+n}$ have intensity $-\beta$. Sources with positive intensity perform points where the degree of birth prevails over the degree of death and it's the opposite in sources with negative intensity.

Proposition. The amount of eigenvalues $\lambda > 0$ of the evolutionary operator $\mathcal{H}_\beta$ (counting their multiplicity) does not exceed the amount of branching sources with positive intensity, and the maximal of these eigenvalues is simple.

The evolutionary operator in this case has form

$$\mathcal{H}_\beta = \mathcal{A} + \beta\Delta_{x_1} + \beta\Delta_{x_2} + \cdots + \beta\Delta_{x_p} - \beta\Delta_{x_{p+1}} - \beta\Delta_{x_{p+2}} - \cdots - \beta\Delta_{x_{p+n}}$$

Assume $|x_i - x_j| = s, i \neq j$. Then $\lambda > 0$ is an eigenvalue of the operator $\mathcal{H}_\beta$ if and only if

$$\begin{aligned}(\beta G_\lambda - \beta G_\lambda(s) - 1)^{p-1}(\beta G_\lambda - \beta G_\lambda(s) + 1)^{n-1} \\ \times ((\beta G_\lambda)^2 + (p+n-2)\beta^2 G_\lambda G_\lambda(s) - (p+n-1) \\ \times (\beta G_\lambda(s))^2 + (p-n)\beta G_\lambda(s) - 1) = 0\end{aligned}$$

For $p \geq 2$ it has two positive roots (of multiplicity $p-1$ and 1):

$$\beta_1 = \frac{1}{G_0 - G_0(s)}, \quad \beta_2 = \max\{A - B, A + B\},$$

where

$$A = \frac{(n-p)G_0(s)}{2(G_0 - G_0(s))(G_0 + G_0(s)(n+p-1))},$$

$$B = \frac{\sqrt{(n-p)^2(G_0(s))^2 + 4(G_0 - G_0(s))(G_0 + G_0(s)(n+p-1))}}{2(G_0 - G_0(s))(G_0 + G_0(s)(n+p-1))}.$$

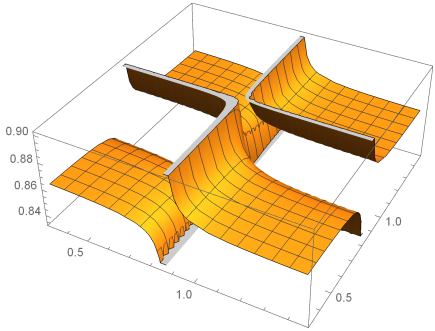

Figure 1. β_1, β_2 and β_3 areas

Example. We consider branching random walk with three sources, that are located in the vertices of a simplex on $\mathbb{Z}^3$. The evolutionary operator has the form

$$\mathcal{H}_\beta = \mathcal{A} + \beta_1 \Delta_{x_1} + \beta_2 \Delta_{x_2} + \beta_3 \Delta_{x_3}.$$

We denote the transition probability of a random walk by $p(t, x, y)$. This function is determined by the transition intensities $a(x, y)$. Green's function of the operator $\mathcal{A}$ can be represented as the Laplace transform of the transition probability $p(t, x, y)$:

$$G_\lambda(x, y) = \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} \frac{e^{i(\theta, y-x)}}{\lambda - \phi(\theta)} d\theta, \qquad \lambda > 0,$$

where $\phi(\theta) = \sum_{z \in \mathbb{Z}^3} a(z) e^{i(\theta, z)}, \theta \in [-\pi, \pi]$.

Denote $G_0(0, 0) = G_0$ and assume $|x_1 - x_2| = |x_1 - x_3| = |x_2 - x_3| = s$. In this case $\lambda > 0$ is an eigenvalue of the operator $\mathcal{H}_\beta$ if and only if

$$\beta_1\beta_2\beta_3(3G_0^2(s)G_0 - 2G_0^3(s) - G_0^3) + (\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)(G_0^2 - G_0^2(s)) - (\beta_1 + \beta_2 + \beta_3)G_0 + 1 = 0.$$

Depending on the values of β_1, β_2 and β_3 this equation can have zero, one, two or three positive solutions (eigenvalues) λ. (see Fig. 1)

3. Conclusions

Models with branching sources located at the simplex vertices (intensities are arbitrary, positive or negative) were presented. In the case of arbitrary number of the branching sources with positive intensities β or negative intensities $-\beta$ we found the upper limit of the amount of eigenvalues $\lambda > 0$ of the evolutionary operator $\mathcal{H}_\beta$ and stated the simplicity of the maximal eigenvalue.

The condition that the highest positive eigenvalue λ_0 of the operator $\mathcal{H}_\beta$ is strictly positive means the exponential growth of the first moment of the total number of particles both at an arbitrary point and on the entire lattice (see [?]): denote by $\mu_t(y)$ the number of particles at site $y \in \mathbb{Z}^d$ and by μ_t the total number of particles at time t, $m_n(t,x,y) := \mathsf{E}_x \mu_t^n(y)$ and $m_n(t,x) := \mathsf{E}_x \mu_t^n$, where E_x stands for the expectation under the condition $\mu_0(\cdot) = \delta_x(\cdot)$. For all $n \in \mathbb{N}$ and $x, y \in \mathbb{Z}^d$, if

$$m(n,x,y) = \lim_{t\to\infty} \frac{m_n(t,x,y)}{m_1^n(t,x,y)}, \quad m(n,x) = \lim_{t\to\infty} \frac{m_n(t,x)}{m_1^n(t,x)},$$

then

$$\lim_{t\to\infty} \mu_t(y) e^{-\lambda_0 t} = \xi\psi(y), \quad \lim_{t\to\infty} \mu_t e^{-\lambda_0 t} = \xi,$$

where $\psi(y)$ is the eigenfunction corresponding to the eigenvalue and ξ is a nondegenerate random variable, are valid for multiple sources in the sense of moment convergence.

Constructed supercritical branching random walk models with arbitrary intensity sources and positive or negative intensity sources presents the approach for finding critical β values.

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